A Historical Survey of Algebraic Methods of Approximating the Roots of Numerical Higher Equations up to the Year 1819

By

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Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Faculty of Philosophy Columbia University

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CONTENTS

I

INTRODUCTION

The feature of mathematics which we may call scientific approximation did not receive the recognition among the ancient Greek geometers that has been accorded it by mathematicians of modern times. Since their work in pure mathematics was based on geometry, the Greek scholars evaluated irrational roots as readily as rational roots by the construction of lines. But when it came to applied mathematics, numerical approximation was often not only desirable but absolutely necessary; and so we have some remarkably close approximations by Archimedes (*c.* 225 B. C.) in his work in mechanics, by Heron of Alexandria (*c.* 200 A. D.) in his work as surveyor, and by Theon of Alexandria (*c.* 375 A. D.) in his astronomical computations. Archimedes even used numerical approximations in the domain of pure geometry in his *Measurement of the Circle.* On the whole, however, the Greek mind did not take to this form of mathematical activity.

The Hindus and the Arabs used approximation methods more extensively than the Greeks, and notable work was done by Brahmagupta (*c.* 510 A. D.), Bhāskara (*c.* 1150 A. D.), and an unknown Arab computer mentioned by Chelebî. However, it was during the Renaissance, when the newly discovered general formula for solving the cubic was found to be inoperative for the "irreducible case," that approximation methods became a vital problem, and for two centuries the keenest mathematical minds worked to find smooth, effective methods for approximating cube and higher roots.

The purpose of this research is to trace the history of the different methods of approximating roots of numerical higher equations that were used up to 1819, the date of the publication of Horner's method; we purpose to trace their early beginnings in finding the roots of numbers and in solving incomplete equations, and to watch their growth into systematically developed general methods for solving complete equations of any degree. Pure trial and error methods, like some of the methods of the Egyptians and, in modern times, that of Junge, will not be taken up, nor shall we discuss such ephemeral methods as those of Fontaine, Collins, and some of

DeLagny's. We shall, moreover, limit the investigation to algebraic methods in algebraic equations of one unknown, and only incidentally shall we touch upon transcendental equations and geometric and trigonometic methods.

In searching for literature on this subject the author found only two articles, outside of the encyclopedias, in the nature of surveys: one by Augustus De Morgan in the *Companion to the British Almanac* (1839), entitled "Notices of the Progress of the Problem of Evolution" and dealing with the extension of the Hindu method of evolution to the solution of numerical equations by Vieta and Horner; and a more extended discussion by F. Cajori in an article entitled "A History of the Arithmetical Methods of Approximation to the Roots of Numerical Equations of one Unknown Quantity," in the *Colorado College Publication* (Colorado Springs, Colorado) for 1910. In the present survey we purpose to study the origin and growth of the various methods by bringing out the illustrations and the details of explanation as far as possible from the original sources.

The writer has had unusual opportunities for studying such source materials in four notable libraries,—the Columbia University Library, the Teachers College Library, the New York City Public Library, and the private library of Professor David Eugene Smith, with its unique collection of editions of the Renaissance and early post-Renaissance periods. By this means he has had access to the early editions and standard translations of Euclid, Archimedes, Heron, Theon, Brahmagupta, Mahāvīrā, Bhāskara, Omar Khayyam, as well as the latest brochures and translations of Chinese and Japanese writers on mathematics. He has also been able to consult such works as those of Leonardo of Pisa, Chuquet, Pacioli, Bombelli, Cardan, Stevin, Pitiscus, Vieta, Oughtred, Hume, Hérigone, Wallis, Newton, Raphson, Colson, Rolle, DeLagny, Halley, Taylor, Waring, Euler, Lagrange, and many others who wrote upon the subject previous to Horner.

II

PRE-ALGORITHMIC METHODS

1. EGYPTIAN AND BABYLONIAN WORK IN FINDING ROOTS

We formerly thought that the root concept originated among the Greeks, but recent discoveries show us that the finding of roots of numbers and the solution of quadratic equations, and even the approximation methods employed in this work, go as far back as the early Egyptian and Babylonian civilizations.

On the Senkereh tablets (*c.* 2000 B. C.) are tables of squares and cubes, which shows that the Babylonians had at least an indirect notion of square and cube roots.[1] Three quadratic equations are known to have been studied in this period. The first one was made known when Griffith (1897) published the mathematical papyrus found by Petrie in Kahun.[2] It deals with areas and requires the solution of the equations $xy = 12$ and $x : y = 1 : \frac{3}{4}$, stated, of course, in rhetorical form. In 1900 Schach discovered in a Berlin papyrus a second problem in quadratics, requiring the solution of the equations $x^2 + y^2 = 100$ and $x : y = 1 : \frac{3}{4}$. The third equation was found in the Kahun papyrus by Schach in 1903. It requires the solution of the equations $x^2 + y^2 = 400$ and $x : y = 2 : 1\frac{1}{2}$. The ancient mathematician solved it by letting $x = 2$, $y = 1\frac{1}{2}$; this gives $x^2 + y^2 = 6\frac{1}{4}$; since $\sqrt{6\frac{1}{4}} = 2\frac{1}{2}$ and $2\frac{1}{2} = \frac{1}{8}$ of 20, he found that $x = 2 \cdot 8 = 16$ and $y = 1\frac{1}{2} \cdot 8 = 12$.

The same method, commonly called the method of False Position, is also used in solving the other two equations.

No irrational roots occur in these equations. But there are indications [3] that the Egyptians had a definite way of approaching the square root of non-square numbers. However, they seem to have been unaware of their irrational quality.

2. EARLY CONCEPTS OF HIGHER ROOTS AND IRRATIONALS AMONG THE GREEKS

That some roots are irrational was first recognized by the Greeks. From measuring the areas of commensurable squares and rectangles

[1] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I (hereafter designated Cantor, I), Leipzig, 1907, p. 28.

[2] M. Simon, *Geschichte der Mathematik im Altertum*, Berlin, 1909, pp. 41–42.

[3] Simon, *Geschichte*, pp. 43–53.

in terms of unit squares they proceeded to find the areas of non-rectangular and irregular figures, such as the triangle and the circle. This they did by finding an equivalent square (Quadrature). Since they neither possessed algebraic symbols nor used the equational form of statement, they found the unknown side by geometrical construction. Some of the sides were discovered to be incommensurable quantities. As early as 525 B. C. Pythagoras recognized the diagonal length of the unit square to be an irrational number. Theodorus of Cyrene (*c.* 410 B. C.) and Plato (*c.* 380 B. C.) discussed the irrational nature of the expressions $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, . . . $\sqrt{17}$. In Book X of his *Elements* Euclid (*c.* 300 B. C.) gave a comprehensive exposition of the theory of irrationals. Up to the time of Archimedes square roots were almost universally evaluated by geometrical constructions; by him they were often evaluated numerically by a process of approximation.

The Greeks seem also to have been the first people to conceive directly the notion of cube root.[4] This also arose from geometrical problems, and, like the square root, was approached geometrically. According to the familiar story the god's demand of an altar twice as large as the old one necessitated the solution of the equation $x^3 = 2a^3$. This was probably solved by Menaechmus (*c.* 365 B. C.) by the aid of conic sections. Archimedes' famous problem of the section of the sphere led to an equation of the type $x^3 - ax^2 + b = 0$, which was also handled geometrically. Heron of Alexandria (*c.* 200 A. D.) was the first one to make a practise of evaluating cube roots numerically; this he did in the course of his practical work by a process of approximation.

3. THE APPROXIMATIONS OF ARCHIMEDES AND HERON OF ALEXANDRIA

In his *Measurement of the Circle* Archimedes (287–212 B. C.) gives among others the following approximations:[5]

$$\frac{1351}{780} > \sqrt{3} > \frac{265}{153}; \quad \sqrt{349450} > 591\frac{1}{8}; \quad \sqrt{5472132\tfrac{1}{16}} > 2339\frac{1}{4}.$$

Nowhere has he disclosed his process, and many conjectures and many ingenious restoration formulas have been advanced as ex-

[4] Tropfke, *Geschichte der Elementar-Mathematik* I (1), Leipzig, 1902, pp. 158–61; 207–10; 223–24.

[5] Cantor, I (1907), p. 316.

planations.[6] We have one bit of information. In discussing Heron of Alexandria (*c.* 200 A. D.), who lived four centuries later, Eutocius tells us that he found square and cube roots by the same method as Archimedes. That remark, together with testing for certain reconstruction formulas, seems to indicate that at times he used methods of Double False Position,[7] as in evaluating the expression $\sqrt{2}$, and at other times he used the formula $\sqrt{N} = \frac{1}{2}\left(a + \frac{N}{a}\right)$, as in finding the square root of 3.

Now, we have Heron's processes described in his own words. A tenth century manuscript of Heron's *Metrica* was recently discovered by Dr. R. Schöne in the Serailo Library in Constantinople. In describing how to find the square root of 720, the area of a certain triangle, Heron says:

"Now, since 720 has no rational square root, we shall find the root differing from it by a very small error. Since the next larger square to 720 is 729 and its side is 27, we divide 720 by 27. The result is 26⅔. To this add 27; that gives 53⅔. One-half of this is 26 ½ ⅓. Therefore the next root of 720 is 26 ½ ⅓. Multiplying 26 ½ ⅓ by itself gives 720 1/36, so that the error amounts to only 1/36 of unity. Should we wish, however, to obtain an error smaller than 1/36, we may put in place of 729, the value we have just found, 720 1/36, and if we do this, we shall find an error much smaller than 1/36."

For want of a name we shall call this the "method of averages." If a is a rational root of N, then $N \div a = a$. But if it is an approximation, and a little too large, as in Heron's example, then $N \div a = a_1$, a little smaller than the exact root. Heron takes as the final approximation $\frac{1}{2}(a + a_1) = \frac{1}{2}\left(a + \frac{N}{a}\right)$.

For approximating the cube root[8] Heron used the method of Double False Position. He explains this process as follows: "We shall now explain how to find the cube root of 100 units. Take the two cubes nearest to 100, one larger, the other smaller. Note how

[6] Cantor, I (1907), pp. 315–17; consult S. Günther, "Quadratische Irrationaliteten der Alten," in *Abhandl. zur Geschichte der Math.*, Vol. IV.

[7] G. Wertheim, "Heron's Ausziehung der irrationalen Kubikwurzeln," in *Zeitschrift für Math. u. Phys.*, XLIX (1899), Hist. Lit. Abt., pp. 1–3.

[8] Curtze, in *Zeitschrift für Math. u. Phys.*, XLII (1897), p. 117–119; Wertheim, in XLIV (1899), pp. 1–3; *Bibliotheca Mathematica* (hereafter called *Bibl. Math.*), third series, VIII (1907–1908), p. 412; Cantor, I (1907), p. 374.

much larger the first one is, namely 25, and how much smaller the second one is, namely 36. Then multiply 36 by 5; the result is 180. To this add 100, which gives 280 [and divide 180 by 280]; the result is $^9/_{14}$. Add this to the root of the smaller cube, namely 4; the result is 4 $^9/_{14}$. Which is the value of the cube root of 100 units as nearly exact as possible."

The ambiguity of the numbers in this example (5 may be $\sqrt[3]{125}$ and also $\sqrt{125-100}$, etc.) excludes a precise knowledge of Heron's process; but it seems to differ from those used by the medieval writers. Curtze, Wertheim, Cantor, and Eneström have all worked out different reconstruction formulas for Heron's process for approximating cube roots.

4. THEON'S METHOD OF EXHAUSTION

The first instance in extant writing of a method of finding square root similar to our present method is found in the writings of Theon of Alexandria (*c.* 365 A. D.). He gives a very full description of how to evaluate irrational roots in sexagesimal fractions.[9] It is a geometrical process referred to a figure. In it he applies a process of exhaustion to Euclid's geometrical counterpart of the expression $(a + x)^2 = a^2 + 2ax + x^2$. This is followed by a generalizing rule. Expressed in modern analytical form, if $a^2 + b = (a + x)^2 = a^2 + 2ax + x^2$ and a is a first approximation, then $x = \frac{b}{2a}$ approximately, and $\sqrt{a^2 + b} = a + \frac{b}{2a}$, approximately. In his illustrative example he finds the square root of 4500° to be 67° 4′ 45″.

Diophantus did brilliant work in approximation, but this was chiefly in connection with inequalities and indeterminate equations, both of which are outside of our topic.

5. SUMMARY

1. We find the first concept of roots in Egypt or in Babylon; in Egypt we also find the solution of quadratic equations by single false position.

2. The concepts of irrationality and of higher roots are first found among the Greeks. Theirs are also the earliest records of a scientific notion of limit and of scientific approximation.

[9] Gow, *History of Greek Mathematics*, Cambridge, 1884, p. 55.

3. We find among them the first recorded use of three approximation methods, namely, the methods of averages and of double false position, by Archimedes and Heron, and the method of exhaustion by Theon. Theon's method differs from Heron's methods in giving sharper results in the limit case; the method of averages and the method of double false position each give two variable limits. Though differing in principle and operation, Heron's method of averages and Theon's method of exhaustion give the same results at each step. For, if $N = a^2 + b$, then Theon's $\sqrt{N} = \sqrt{a^2 + b} =$ $\frac{1}{2}\left(a + \frac{N}{a}\right) = \frac{1}{2}\left(a + \frac{a^2 + b}{a}\right) = a + \frac{b}{2a}$.

4. The existence of a root and never more than one root was taken for granted among both Egyptians and Greeks.

III

ALGORITHMIC METHODS FOR APPROXIMATING THE ROOTS OF PURE POWERS

1. THE HINDU METHOD OF EXHAUSTION

The superior notation and convenient symbolism of the Hindus enabled them to use processes of inversion to a degree impossible to Theon. In fact that became the favorite line of attack in much of their mathematics. Square and cube roots were found by Āryabhaṭa[1] (*b.* 476 A. D.) from the inverse of the formulas $a^2 + 2ab + b^2$ and $a^3 + 3a^2b + 3ab^2 + b^3$.

When the Hindu decimal notation with its place values came into use, the work of finding roots was so arranged that an application of the inversion process would evolve the digits of the root in order (whence "evolution" in texts on arithmetic and algebra). To effect this the number was divided into periods and the work arranged in columns and lines. This "Hindu method," as it was often called in the Middle Ages, is described for us by Brahmagupta (*b.* 598 A. D.).[2] Śrīdhara,[3] in his *Triśatikā* (1040 A. D.) gives the rules in full, as also does Bhāskara (*b.* 1114 A. D.).[4] The following is a free translation from Bhāskara, by Taylor:

"The first place on the right is called ghana or cube; the two next places aghana or not-cube.—Subtract the cube contained in the final period from the said period; put down the root *of the cube* in a separate line, and after multiplying its square by three, divide the antecedent figure by the result, and write down the quotient in the separate line: Then multiply the square of the quotient by the preceding number *in that line* and by three, and after subtracting the product from the next antecedent figure cube the said quotient, and subtract the result from the next antecedent figure. Thus repeat the process through all the figures. The separate line contains the Cube Root."

[1] Cantor, I (1907), p. 625.

[2] H. T. Colebrooke's translations of Brahmagupta and Bhaskāra, London, 1817, pp. 279–81. Hereafter referred to as Colebrooke.

[3] Kaye, on *Tristika*, in *Bibl. Math.* (3) XIII (1912–13), p. 209.

[4] Taylor's translation of *Lilawati*, Bombay, 1816, p. 20.

[illegible]

The Hindu method was adopted by the Arabs [illegible]

[illegible] From the Arabs [illegible]

[illegible]

[illegible] begins with [illegible]

[illegible] cian and an astronomer [illegible]

Europe [illegible]

LaRoche [illegible] and Gemma Frisius [illegible]

[illegible]

[illegible]

The Hindu [illegible]

[illegible]

As far back as [illegible]

[illegible]

Among the Arabs [illegible]

[illegible]

[illegible]

and al-Qalasâdî (*c.* 1475).[10] This rule was also used by the Jewish writer Johannes Hispalensis (*c.* 1140). Al-Karchî (*c.* 1020) used a different formula, namely $\sqrt{a^2 + b} = a + \dfrac{b}{2a + 1}$. How he derived this formula is not known; possibly he used a method of double false position and interpolation by proportional parts similar to the one used later by Leonardo of Pisa for cube roots. Ibn Albannâ (*c.* 1300) used al-Karchî's formula for $b > a$, but Theon's formula for $b \leq a$. Leonardo used Theon's formula with a corrective supplement, as follows: $\sqrt{a^2 + b} = a + \dfrac{b}{2a} - \dfrac{\left(\frac{b}{2a}\right)^2}{2\left(a + \frac{b}{2a}\right)}$. A formula frequently used in the Middle Ages was $\sqrt{A} = \dfrac{1}{a}\sqrt{Aa^2}$. In the Renaissance period Bombelli (1572), Cataldi (1613), and Schwenter (1618) approximated irrational square roots by the use of continued fractions.

A formula for approximating irrational cube roots was invented by Leonardo of Pisa.[11] In deriving this he used double false position and interpolation by proportional parts. He lets a^3 and $(a + 1)^3$ be the two cubes nearest to $a^3 + b$. Then, since $a^3 < a^3 + b < (a + 1)^3$, we have $0 < b < 3a(a + 1) + 1$. This relation gives a criterion for the proper choice of a. Also, if we increase a by unity we increase the resulting cube by $3a(a + 1) + 1$. What increase in a will increase the resulting cube by b? By proportion he obtains as an answer $\dfrac{b}{3a(a + 1) + 1}$. Therefore $\sqrt[3]{a^3 + b} = a + \dfrac{b}{3a(a + 1) + 1}$.[12] He gives these illustrations of his formula:

[10] Günther, *Quadratische Irrationaliteten der Alten*, pp. 45–46; Cantor, I (1908), pp. 633, 641; Hankel, *Zur Geschichte der Mathematik im Alterthum und Mittelalter* (hereafter referred to as Hankel), Leipzig, 1874, p. 185.

[11] Boncompagni, *Leonardo Pisano*, I, pp. 370, 378, 380–81; Cantor, II (1900), pp. 31–32; *Bibl. Math.* (3) II (1902), pp. 350–54.

[12] Juan de Ortega (1512) used a modification of this formula, viz., $\sqrt[3]{a^3 + b} = a + \dfrac{b}{3a(a + 1)}$. See Cantor, II (1900), p. 388.

$\sqrt[3]{900} = 9 + \frac{171}{271} = 9\frac{2}{3}$ approximately, and $\sqrt[3]{2345} = 13 + \frac{148}{547}$ $= 13\frac{1}{4}$, approximately.

To find the roots of large irrational numbers two plans were followed, the Hindu arrangement being used in both cases. Leonardo illustrates them both.[13] In one of his solutions, $\sqrt{927435} = 963 + \frac{11}{321} - \frac{\left(\frac{11}{321}\right)^2}{2\left(962 + \frac{11}{321}\right)}$, the derivation of 963 is shown as follows:

		5	8	7	
9	2	7	4	3	5
			9	6	3
			9	6	3

We find this method used as late as the seventeenth century in the *Kholasat-al-Hisâb* [14] of Behâ Eddîn (*c.* 1600). Another plan is followed by Leonardo in this solution:

$$\sqrt{7234} = \frac{1}{100}\sqrt{72340000} = \frac{1}{1000}.\ 8505\frac{1}{4} = 85\ \frac{1}{20}\ \frac{1}{400}.$$

This plan became increasingly popular and when in 1539 a man of Cardan's prestige adopted it systematically in his arithmetic,[15] accompanied by clear and concise rules, it became the standard method in Europe.

It had taken a thousand years, from Āryabhaṭa to Cardan, to perfect this approximation method. But it is the most effective and most widely used method known. The methods of Vieta and of Horner are its lineal descendants.

[13] Boncompagni, *Leonardo Pisano*, I, p. 355; Cantor, II (1900), p. 30; Curtze, "Ueber eine Algorismus Schrift des XII. Jahrhunderts" in *Abhandl. Geschicht. Math.*, VIII (1898), pp. 1–28.

[14] Taylor's *Lilawati*, Preface, p. 14; Hutton's *Tracts* II, London, 1812, p. 180. A manuscript copy of the *Kholasat-al-Hisâb* is found in Professor D. E. Smith's private collection.

[15] Cardan's Arithmetic, chap. 23.

3. SUMMARY

1. The superior symbolism and notation of the Hindus enabled them to invent and use with ease formulas based on the principle of inversion.

2. For finding the rational roots of numbers too large for handling by inspection they used a method of exhaustion. It was based on the inversion of the expressions $a^2 + 2ab + b^2 = (a + b)^2$ and $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$, and on the decimal notation with its place values. Its essence was the evolving of the digits of the root by orders (Evolution). This method was also used for irrational roots later on, and European scholars standardized an adaptation of it for approximating irrational roots by decimal fractions.

3. The five principal approximation formulas invented and used in the Middle Ages were:

$$(a)\quad \sqrt{a^2 + b} = a + \frac{b}{2a}; \qquad (b)\quad \sqrt{a^2 + b} = a + \frac{b}{2a + 1};$$

$$(c)\quad \sqrt{A} = \frac{1}{a}\sqrt{Aa^2}; \qquad (d)\quad \sqrt[3]{a^3 + b} = a + \frac{b}{3a(a + 1) + 1};$$

$$(e)\quad \sqrt[3]{a^3 + b} = a + \frac{b}{3a(a + 1)}.$$

IV

APPROXIMATIONS FOR SPECIFIC PURPOSES

I. THE RISE OF NUMERICAL HIGHER EQUATIONS AMONG THE ARABS

Although the Arabs did not contribute much original matter to algebra they vitalized it and enriched its contents by applying algebraic operations to the problems of Greek geometry and to their own problems in astronomy and trigonometry. This led them directly to numerical higher equations. Archimedes's problem of the section of the sphere, which led to a cubic, was first attempted by al-Mâhânî (*c.* 860), and was later solved successfully by al-Khâzin (*c.* 950) with the aid of conic sections. The trisection of angles and the computation of sides of regular polygons led to other cubics,[1] which were also solved by conic sections; for they soon came empirically to the conclusion that cubics could not be solved algebraically. The solution of numerical cubic equations by intersecting conics was the greatest original contribution to algebra made by the Arabs. These solutions remained unknown to the Western world, and were rediscovered in the seventeenth century by Descartes, Thomas Baker and Edmund Halley. The success of the Arab scholars in this field may have deterred them from trying methods of approximation. What they might have done in this field may be inferred rather than judged from the solitary example left us in Saracen writings.

2. A SOLUTION BY AN UNKNOWN ARAB SCHOLAR

In Mîram Chelebî's annotations of Ulugh Beg's astronomical tables (1498) there is explained a method of solving the equations $Px = x^3 + Q$, where P and Q are positive and x^3 is small in comparison with Q.[2] Chelebî explains that this equation was used for a specific purpose, namely, to find the sine of 1°. With the Persian astronomers, as with us, the ordinary trigonometric interpolation

[1] *L'algèbre d'Omar Alkayyami* (translated and edited by Woepcke), Paris, 1851, pp. 54–57; p. 82; Hankel, p. 277.

[2] Cantor I (1907), pp. 781–82; Hankel, pp. 289–93; Braunmühl, *Geschichte der Vorlesungen der Trigonometrie* I, Leipzig, 1900–3, pp. 72–74.

would fail for sin 1°. The solution is commonly attributed to Giyât Eddîn al-Kâschî. Braunmühl, however, thinks it is due to al-Zarkâli (*c.* 1080).

In this particular equation $P = 45^r$ and $Q = 47^r\ 6^p\ 8'\ 29''$, where the radius length 1^r is the sexagesimal unit and is equal to 60^p. The solution is as follows:

From the equation,

$$x = \frac{Q + x^3}{P}; \text{ for a relatively small } x \text{ we have } x = \frac{Q}{P}.$$

$$\text{Then } x = a + r;\ a + r = \frac{Q + (a + r)^3}{P} = a + \frac{R + (a + r)^3}{P}$$

$$= a + \frac{R + a^3}{P}.$$

Suppose $\dfrac{R + a^3}{P} = b$, with remainder S; that gives $R = bP + S - a^3$.

$$\text{Then } x = a + b + s;\ a + b + s = \frac{Q + (a + b + s)^3}{P}$$

$$= a + \frac{R + (a + b + s)^3}{P} = a + \frac{R + (a + b + s)^3}{P}$$

$$= a + b + \frac{S - a^3 + (a + b + s)^3}{P} = a + b + \frac{S + (a + b)^3 - a^3}{P}$$

Suppose the last fraction equals c, with a remainder T.

Then $x = a + b + c + s$, and so on.

The actual problem is to evaluate sin 1° from the known trigonometric relation $60^r \sin 3° = 3.60^r \sin 1° - 4 \sin^3 1°$. Letting $x = \sin 1°$ and simplifying, the equation takes the form $45^r\, x = x^3 + \frac{1}{4} \cdot 60^r \sin 3°$. The Arabs knew the last term to equal $47^r\ 6^p\ 8'\ 29'' = Q$.

The following [3] is Chelebî's problem, carried out to seconds; in the Arab text it is carried out to fourths.

[3] Hankel, p. 292.

Divisor	P	45^r	a	b	c
Quotient	x		1^p	$2'$	$49''$
Dividend	Q	47^r	6^p	$8'$	$29''$
	aP	45			
	R	2			
	A^3		1		
	$R \quad a^3$		127		
	bP		90		
	S		37		
	$(a+b)^3 - a^3$			6	12
	$(a+b)^3 - a^3$			2234	41
	cP			2205	
	T			29	

That the method was effective for its particular purpose is shown by Ulugh Beg's tables. That it had its limitations is also obvious: it is laborious; and it is restricted to equations where P and Q are positive and x is relatively small. Yet the advance in the theory of equations was soon to remove the latter limitations; and refinements and modifications by men like Harriot, Raphson, and Halley might have reduced the former. Certainly this remarkable solution was the feat of some genius seemingly unsupported and isolated.

We quote Hankel's appreciation:

"This beautiful method of solving numerical equations stands second to none of the methods of approximation invented in the Occident since the time of Vieta in fineness and elegance. Apart

from the solutions of square and cube roots, with which it has a similarity of principle, it is the first method of successive numerical approximations which we meet in the history of mathematics."

3. A SOLUTION BY LEONARDO OF PISA

The last statement may or may not be true. There were at least two other modes of approximation in the thirteenth century. One was a well-developed system in China which we shall describe in the chapter on Horner's method. The other was the method employed by Leonardo of Pisa.

At a scientific tournament in Pisa in 1225 he was asked by the Emperor Frederick II to solve the equation $x^3 + 2x^2 + 10x = 20$ by Euclid's method of ruler and compasses.[4] After several unsuccessful attempts he changed his method of attack and showed by rigorous analysis that the roots of this equation could not be found by ruler and compasses. Then he finally gave this approximate solution: $x = 1^\circ\ 22'\ 7''\ 42'''\ 33^{\text{iv}}\ 4^{\text{v}}\ 40^{\text{vi}}$. What process he used no one knows. But after seven centuries of further study in calculation we differ from his result by only $1\frac{1}{2}^{\text{vi}}$ or $\dfrac{1}{31{,}104{,}000{,}000}$. As with the solution of the Arab scholar, so with this one: we marvel alike at its isolation and at its accuracy.

Many attempts have been made to explain Leonardo's method. Zeuthen thinks he used a method of false position with interpolation similar to his method for the cube root of numbers. Others think there was some relation between his work and that of the Arab scholar. Professor David Eugene Smith thinks he was acquainted with the work of the Chinese. There is evidence in his writings that he was familiar with the Chinese method of approximating roots of indeterminate equations.[5]

4. SUMMARY

1. The Arabs added little that was original to the science of algebra. But they vitalized it by applying the new science to problems of geometry, astronomy, and trigonometry.

[4] Cantor, II (1900), p. 46; Boncompagni, *Intorno ad alcune opere di L. Pisano*, pp. 17–19.

[5] Cantor, II (1900), p. 26; Curtze in *Zeitschrift für Math. u. Phys.* XLI, Hist. Lit. Abt., pp. 81, 82.

2. This gave rise to higher numerical equations, many of which they solved by intersecting curves. The success of this method undoubtedly delayed the work of solving them algebraically by approximation.

3. We have only two extant algorithmic solutions by approximation from this period, namely that described by Mîram Chelebî and the solution of Leonardo of Pisa. They were both effected for a particular purpose.

V

METHODS INVOLVING DOUBLE FALSE POSITION AND PROCESSES OF INTERPOLATION

1. THE GENERAL NATURE OF THESE METHODS

The first successful general methods of approximating roots in complete equations were those involving "double false position," and the history of these methods is that of incomplete equations repeated. The chief problem in this mode of attack is to discover a practical way of finding intermediate ("mean") values that will converge to the true root. Three chief plans sprang up: the "averaging" rule of Chuquet; the methods of interpolation by proportional parts used by Cardan, Bürgi, and Pitiscus; and the process of orderly evolution of digits used by Stevin, known later as Stevin's Rule.

2. CHUQUET'S RULE OF MEAN NUMBERS

Chuquet's *Triparty* [1] (1484) was little known outside of France. It was circulated only in manuscript, not being printed until 1880. Yet an examination of the textbooks of LaRoche, Petro Sanchez Ciruelo,[2] and other writers reveals that he had considerable influence in France. His was a mathematical mind of a high order. His symbolism and arrangement of work were much ahead of his time. In the theory of equations he held advanced views; he recognized negative and hinted at imaginary roots, and gave a definite statement for the law of signs.

In his Rule of Mean Numbers he states, without proof, that $\dfrac{a_1 + a_2}{b_1 + b_2}$ lies between $\dfrac{a_1}{b_1}$ and $\dfrac{a_2}{b_2}$. Also, if b_1 and b_2 have the same signs, the following differences will have the same sign:

$$\frac{a_1}{b_1} - \frac{a_1 + a_2}{b_1 + b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1(b_1 + b_2)} \text{ and } \frac{a_1 + a_2}{b_1 + b_2} - \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_2 (b_1 + b_2)}.$$

The method is perfectly general, and he applies it to complete and

[1] *Triparty*, pp. 653–54 in Boncompagni, XIII (1880); Cantor, II (1900), pp. 351, 359–60; Boncompagni, XIV (1881), pp. 416, 419.

[2] Cantor, II (1900), p. 386.

incomplete equations alike. An integer a is put in the fractional form $\frac{a}{1}$. For rational roots the approximation terminates of itself; for irrational roots the rapidly increasing denominator insures a more rapid convergence than the arithmetic mean that had been used by others.[3] As an illustration we may take his solution of the equation $x^2 + x = 39\frac{13}{81}$.

In this case we see that $x = \frac{5}{1}$ is too small, while $x = \frac{6}{1}$ is too large. Hence the first mean is $\frac{5+6}{1+1} = \frac{11}{2}$. By substitution this proves too small. Therefore the new limits are $\frac{11}{2}$ and $\frac{6}{1}$, the new mean being $\frac{11+6}{2+1} = \frac{17}{3}$. This second mean proves too small. The limits $\frac{17}{3}$ and $\frac{6}{1}$ give the third mean $\frac{17+6}{3+1} = \frac{23}{4}$. This is also too small. The limits $\frac{23}{4}$ and $\frac{6}{1}$ give $\frac{23+6}{4+1} = \frac{29}{5}$, which proves too large. The limits $\frac{23}{4}$ and $\frac{29}{5}$ give the mean $\frac{23+29}{4+5} = \frac{52}{9}$, and this proves to be the exact root.

3. THE REGULA AUREA OF CARDAN

The first general method of approximating roots of numerical higher equations to be printed and publicly known was the "Regula Aurea" (Golden Rule) of Cardan (1501–1576) published in his *Ars Magna* [4] (1545), six years after the publication of his rules for square and cube roots. It is built on the basis of two false positions and a particular mode of interpolation. Though he uses it only for equations of the third and fourth degree, it is applicable to equa-

[3] An adaptation, by no means an improvement, of Chuquet's method was worked out by DeLagny in his Method of Mediation (1697).

[4] Chapter XXX, entitled "De regula aurea"; Cantor, II (1900), pp. 505–507.

tions of every degree. He restricts the rule to equations where the coefficients are positive or zero.

The following is his derivation of the rule, presented in modern analytical form:

Let $f(x) = k$, where $f(x)$ is a polynomial in x arranged in descending order. Let a and $(a + 1)$ be two positive integers such that $f(a) = k - b$ and $f(a + 1) = k + b'$. Then the root x lies between a and $a + 1$. That is, $x = a + t(\overline{a + 1} - a)$ or $x = (a + 1) - e(\overline{a + 1} - a)$, where the corrections t and e are positive proper fractions. The first may be called the additive, the second the subtractive form. Since $f(a + 1) > f(x) > f(a)$, we have $f(a + 1) - f(a) > f(x) - f(a) > 0$ and consequently

$$\frac{f(x) - f(a)}{f(a + 1) - f(a)} = \frac{k - (k - b)}{(k + b') - (k - b)} = \frac{b}{b + b'} < 1.$$

Take $\dfrac{b}{b + b'}$ as an approximate value of t. Then $x = a + \dfrac{b}{b + b'}$.

Since $\dfrac{b}{b + b'} < 1$ our next additive form will then be $f\left(a + \dfrac{b}{b + b'}\right)$ $= k - b''$.

Placing this new additive value in the subtractive form we have

$$x = (a + 1) - e\left\{(a + 1) - \left(a + \frac{b}{b + b'}\right)\right\} = a + 1 - \frac{b'\,e}{b + b'}.$$

In the subtractive form e is less than unity. For an approximate value set $e = \dfrac{f(a + 1) - f(x)}{f(a + 1) - f\left(a + \dfrac{b}{b + b'}\right)} = \dfrac{(k + b') - k}{(k + b) - (k - b'')} = \dfrac{b'}{b' + b''}$. This gives the third approximation $x = a + 1 - \dfrac{b'}{b + b'}$. $\dfrac{b'}{b' + b''} = r$, say. According as k lies between $f(r)$ and $f(a)$ or between $f(r)$ and $f(a + 1)$ one must, in continuing, use the additive or the subtractive process.

His first illustrative example is the solution of the equation $x^4 + 3x^3 = 100$, the root of which he finds to be $2\frac{2775}{4697}$. The following is his solution and arrangement:

$$\begin{array}{ccc} 2 & & 3 \\ 40 & 100 & 162 \\ 60 & & 62 \end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{122}$$

$$\frac{60}{122} \;\Big|\; \frac{30}{61} \qquad 2\frac{30}{61} \qquad 162$$

$$77 \;\text{———}\; 85$$

$$62 \;\text{———}\; 100$$

$$2\frac{30}{61} \;\Big|\; 3 \;\Big|\; \frac{31}{61}$$

$$\frac{31}{61} \;\Big|\; 62 \quad \frac{1922}{61} \quad 77$$

$$\frac{1922}{4697} \;\Big|\; 3 \;\Big|\; 2\frac{2775}{4697}$$

Contrary to his own restrictions on the signs of the coefficients Cardan uses the rule in the equation $x^3 = 6x + 20$, with disastrous results.

4. STEVIN'S CONTRIBUTION

The approximation work in Stevin's algebra (1685) might with equal propriety be discussed in connection with Vieta or Newton. For Stevin (1548–1620) anticipated Newton's analytical form and treatment as well as Vieta's idea of evolving the digits of the root

of a complete equation order by order. But since his scheme of deriving the digits involved the successive choice of two positions, we discuss his method in this chapter. We can best explain his process by recording one of his illustrations.[5]

$$
\begin{array}{llll}
 & \text{Suppose } x^3 = 300x + 33915024; & & \\
\text{if } x = & 1, \text{ then} & 1^3 < 300\cdot 1 & + 33915024; \\
\text{if } x = & 10, \text{ then} & 10^3 < 300\cdot 10 & + 33925024; \\
\text{if } x = & 100, \text{ then} & 100^3 < 300\cdot 100 & + 33915024; \\
\text{if } x = & 1000, \text{ then} & 1000^3 > 300\cdot 1000 & + 33915024.
\end{array}
$$

Hence x lies between 100 and 1000. By trying $x = 200$, $x = 300$, $x = 400$, he finds the root to lie between 300 and 400. Similarly trying 310, then 320, then 330, he finds x to lie between 320 and 330. A similar procedure for the units gives 324 as the exact value of the root. Stevin observes that an irrational root can be approximated to within any desired degree of accuracy by using this method and his scheme of computing decimal fractions. This method was advocated and used considerably by Albert Girard in his *Invention Nouvelle en l'Algèbre* (1629). Though laborious, the method is general. "Stevin's rule" was used by later algebraists, like Oughtred (1631), Kersey (1673), and Saunderson, in connection with other methods.[6]

5. THE METHODS OF BÜRGI AND PITISCUS

Pacioli's *Summa* (1494), the earliest printed work on arithmetic and algebra, gave to Renaissance scholars the method of double false position and interpolation by proportional parts used by Leonardo for finding roots of numbers. Leonardo's method now came to be employed frequently also in the solution of higher numerical equations, and Pacioli even used it in an exponential equation.[7] This method, which has always been basic in finding trigonometric functions, was used by Jost Bürgi in solving equations which rose out of his work in trigonometry.[8] We give his solu-

[5] Stevin, *Les Œuvres Mathématiques* (editor Girard, 1634), I, problem 77; Cantor, II (1900), p. 628.

[6] N. Saunderson, *Elements of Algebra*, II, Cambridge, 1790, p. 728.

[7] Pacioli, *Summa*, fol. 186 recto to 188 recto; Cantor, II (1900), pp. 325–26.

[8] Cantor, II (1900), pp. 626, 645–46, 648; C. I. Gerhardt, *Geschichte der Math. in Deutschland*, Munich, 1877, pp. 76–82; 84–87.

tion of such an equation, namely, $9-30x^2+27x^4-9x^6+x^8=0$. The root has been found to lie between 0.68 and 0.69; let $x = 0.68 + h$, and $f(x) = 0$. Then

$f(0.68) = \quad 0.0569$	$f(0.68) = \quad 0.0569$
$f(0.69) = -\underline{0.0828}$	$f(0.68+h) = \underline{0.}$
Difference $0.01 : 0.1397$	Difference $h : 0.0569$

Hence $0.1397 : 0.0569 = 0.01 : h$; or $h = 0.004$, and $x = 0.684$. Repeating the process with $f(0.6840)$ and $f(0.6841)$ he gets $0.00140012 : 0.00056410 = 0.0001 : h$; or $h = 0.00004029$. Hence $x = 0.68414029$.

A more conventional method of double false position is used by Pitiscus (1612), a pupil of Bürgi. In solving the equation[9] $5{,}176{,}380 = 3x - x^3$ he sets $x = 1{,}725{,}460 + \frac{x^3}{3}$. Manifestly $x > 1{,}725{,}460$. By trial he finds x to lie between 1,730,000 and 1,740,000. Let us call these two values a_1 and a_2 and represent the equation by $f(x) = k$. Then $f(a_1) = k - 38{,}158 = k - d_1$; $f(a_2) = k - 9{,}061 = k - d_2$. Then

$$x = \frac{a_2d_1 - a_1d_2}{d_1 - d_2} = 1{,}743{,}114.$$

6. SUMMARY

1. The first successful general method of solving complete numerical equations by approximation was given by Chuquet in his *Triparty* (1484). It was not printed until 1880 and remained comparatively little known.

2. The first printed method was Cardan's Regula Aurea (1545). It was restricted as to the signs of the coefficients but not as to the degree of the equation. It never came into general use.

3. In 1585 Stevin explained a method of evolving the digits of the roots by order.

4. Pacioli and Bürgi extended Leonardo's method of finding cube roots to the solution of complete equations of higher degree. This method and the conventional method of double false position exemplified by Pitiscus were the ones most generally used before the year 1600.

[9] Pitiscus, *Trigonometriae*, pp. 50–53.

VI

VIETA'S EXTENSION OF THE HINDU METHOD OF APPROXIMATION TO COMPLETE EQUATIONS

1. VIETA'S METHOD OF APPROXIMATION

A new epoch in the history of the solution of numerical equations begins with the year 1600, the date of the first publication of Vieta's *De Numerosa Potestatum Purarum atque Adfectarum ad Exegesin Resolutione Tractatus.* In it was explained a systematic and concise way of solving any equation, complete or incomplete, by methodically finding the successive digits of the root, beginning with the highest order. Why no one before Vieta should have thought of applying to the solution of complete equations the Hindu method of finding roots of large numbers may seem strange to us unless we reflect that useful inventions usually seem very natural after they are invented. At any rate, this method had been applied to large numbers for a thousand years, had appeared in definite forms for four hundred years, and had been perfected in its finest details, before any one thought of extending it to complete equations. It was the first comprehensive method of solving such equations that had been attempted, and it involved no restrictions as to terms, signs, or degree, for it applied to a form into which any equation can be changed. Vieta calls an equation "duly prepared" if it has the form $x^n + ax^{n-1} + bx^{n-2} + \ldots\ldots = K$ where K is positive and all the coefficients are integers.

De Numerosa Potestatum was first printed in Paris in 1600 for private circulation. In 1646 Franz van Schooten, of Leyden, again printed it in a one-volume collection [1] of Vieta's works. It contains sixty-six pages, of which the first ten deal with incomplete equations. He arranges the solution in a way slightly different from the conventional one in order the better to show the analogy between the pure and the affected powers.[2] We

[1] *Vietae Opera Mathematica*, Leyden, 1646; hereafter referred to as Vieta.

[2] Vieta classified powers and equations as *pure* and *affected* ("ad-fected"). The names are geometrical in origin; for Vieta considered all the terms in an equation homodimensional. Thus in the "affected" equation $x^3 + 30x = K$, the "resolvend" K is considered three-dimensional; and 30 is the "plane coefficient" in the "adjoined solid"

insert the solution for one pure equation and two types of affected equations,[3] viz.:

(1) $x^3 = 157{,}464$; where $x = 54$.
(2) $x^3 + 30x = 14{,}356{,}197$; where $x = 243$.
(3) $95{,}400x + x^3 = 1{,}819{,}459$; where $x = 19$.

The arrangement is that of Vieta. The explanations, too, are a close paraphrase in modern notation of his marginal and other explanations rhetorically expressed.

(1) Let $f(x) = K$ for $x^3 = 157{,}464$;
if $x = a + b$, then $a^3 + 3a^2b + 3ab^2 + b^3 = 157{,}464$.
"Proponantur 1 C, aequari 157,464.

I. Eductio lateris singularis primi

K : Resolvend	157	464	$(a = 5$
a^3	125		
$K - f(a)$: Residual	32	464	

II. Eductio lateris singularis secundi

$K - f(a)$: Resolvend	32	464	
$3a^2$	7	5	
$3a$		15	
Divisor	7	65	$(b = 4$
$3a^2b$	30		
$3ab^2$	2	40	
b^3		64	
Subtrahend	32	464	

Itaque 1 C, aequetur 157,464 fit 1 N 54."

(2) Let $f(x) = K$ for $x^3 + 30x = 14{,}356{,}197$;
if $x = a + b$, then $(a^3 + 3a^2b + 3ab^2 + b^3) + 30(a + b) = 14{,}356{,}197$.

"Proponantur 1 C + 30 N, aequari 14,356,197.

$30x$. The names "pure" and "affected" continued in use until the latter part of the 19th century. Since, in the modern theory of equations, the basic form is $a_0x^n + a_1x^{n-1} + \ldots + K = 0$, recent writers use *complete* and *incomplete* as being more descriptive. In discussing the works of Vieta and his immediate successors, we shall employ the terminology used by them.

[3] Vieta, pp. 167–68; 177–78; 179.

I. Eductio lateris singularis primi

Plane coefficient		3	0	
K: Resolvend	14	356	197	($a = 2$
a^3	8			
$30a$		6	0	
$f(a)$	8	006	0	
$K-f(a)$: Residual	6	350	197	

II. Eductio lateris singularis secundi

Plane coefficient			30	
$K-f(a)$: Resolvend	6	350	197	
$3a^2$	1	2		
$3a$		6		
Divisor	1	260	30	($b = 4$
$3a^2b$	4	8		
$3ab^2$		96		
b^3		64		
$30b$		1	20	
Subtrahend	5	825	20	
$K-f(a\ b)$		524	997	

III. Eductio lateris singularis tertii

Plane coefficient		30	
Resolvend	524	997	
$3a^2$	172	8	
$3a$		72	
Divisor	173	550	($b = 3$
$3a^2b$	518	4	
$3ab^2$	6	48	
b^3		27	
$30b$		90	
Subtrahend	524	997	

Itaque 1 C 30 N, aequetur 14,356,197 fit 1 N 243."

(3) Let $f(x) = K$ for $95{,}400x + x^3 = 1{,}819{,}459$;
if $x = a + b$, then $95\,400(a + b) + (a^3 + 3a^2b + 3ab^2 + b^3)$
$= 1{,}819{,}459$.

"Paradigma cum solidum adfectionis sub latere majus est cubo.

I. Eductio lateris primi inanis ante devolutionem

Plane coefficient	9	540	0
K: Resolvend	1	819	459

Quoniam 9 major est unitate, fit devolutio.

II. Eductio lateris singularis primi post devolutionem

Divisor		954	00	$(a = 1$
K: Resolvend	1	819	459	
$95\,400a$		954	00	
a^3		1		
$f(a)$		955		
$K - f(a)$: Residual		864	459	

III. Eductio lateris singularis secundi

Plane coefficient	95	400	
$K - f(a)$: Resolvend	864	489	
$3a^2$		3	
$3a$		3	
Divisor	95	730	$(b = 9$
$95\,400b$	858	600	
$3a^2b$	2	7	
$3ab^2$	2	43	
b^3		729	
Subtrahend	864	459	

Itaque si 95,400 N + 1C, aequentur 1,819,459 fit 1 N 19."

In a similar fashion he illustrates the solution of equations of the fourth, fifth, and sixth degrees. Vieta's work involves mainly four points:

1. The first step is to ascertain the number of digits in the root. For pure powers, of course, this is easy. For affected powers there will also be as many digits in the root as there are periods in the resolvend if the highest power is dominant; that is, if the pure power is larger than the adjoined solid or solids. This will generally be indicated by the size of the coefficients. Thus examples (1) and (2) will have roots of 3 digits. A more difficult situation arises if the highest power is not dominant. Thus in example (3) the resolvend has 3 periods, yet the root has only 2 digits. Such a case he calls "devolution." If the cube and the adjoined solid had been of opposite signs there would have been more digits in the root than periods in the resolvend. Such cases are called "anticipations." Vieta and all his followers found these cases full of practical difficulties and he offers many suggestions for reducing and circumventing them. One device is shown in our third illustration.

2. With Vieta the fundamental operation for finding the roots of pure as well as affected equations is the expansion of the powers of a binomial.[4]

3. For each approximation a, Vieta takes the residual $K-f(a)$ as the new resolvend. The entire residual is put down each time.

4. Vieta gets his trial divisors by taking the sum of all the coefficients of the powers of b except the coefficient of b^n, which is invariably unity. Both Cantor and Hankel have given erroneous explanations of Vieta's divisors. The mistake seems to come from representing Vieta's process in the compact equational form of Newton's scheme. At least they attribute to him Newton's divisor. Following Hankel's and Cantor's explanation, the first divisor in Example (1) should be 120,030. Vieta's divisor, however, is 120,630.[5] In this connection it should be said that both these historians give an erroneous impression of the plan of Vieta's method. They lead one to suppose that Vieta's method of evolving digits is the same as Newton's method of substituting values in equations of series; and Hankel says as much. But the two differ in origin and form, and in certain of the steps.

In developing his approximation method Vieta uses for con-

[4] Cantor's exposition of Vieta's process, II (1900), pp. 640–41, where he sets $x = x_1 + x_2 + x_3$, is apt to be misleading. Vieta did not use that substitution, though his expositors Hume (1636) and Hérigone (1642) used it occasionally.

[5] Hankel, p. 370; Cantor, II (1900), p. 641.

venience equations having rational roots. But he adds[6] that irrational roots can be approximated in the same way after first transforming the equation into one whose roots are decimal multiples of the roots of the given equation. This, of course, is nothing but an extension of Cardan's rule for irrational numbers. Thus, in the equation $x^3 = 2$, the root is found from solving $x^3 = 2{,}000{,}000, \ldots\ldots$; likewise if $x^3 + 600x = 8{,}000$, the root correct to hundredths is found by solving $x^3 + 60{,}000x = 8{,}000{,}000$.

2. THE MODIFICATIONS OF VIETA'S METHOD BY HARRIOT

It was not in his homeland that Vieta's method met with the greatest appreciation. The people of the British Isles have always been partial to mathematical processes dealing with numerical calculation. Three English scholars, Thomas Harriot (1560–1621), William Oughtred (1574–1660), and John Wallis (1616–1703), all noted as teachers and writers, became the chief protagonists of the new method.

In his *Artis Analyticae Praxis*[7] (written before 1621 and published posthumously in 1631) Thomas Harriot introduces two devices for expediting the calculation in Vieta's method. One of these is the elimination of the second term; the second is his *Canones Directorii*, a collection of the different cases of numerical solutions, ready and arranged for use under various canonical equations, showing the elements necessary to form the several subtrahends and resolvends. This plan gained much vogue and we find Raphson, Halley, and DeLagny doing the same thing in connection with Newton's method.

In the actual computation he uses Vieta's method unchanged. We find no basis for De Morgan's statement that he formed "only so much of the divisor as is necessary for the determination of the next figure." Nor does Wallis, who is very partial to Harriot, mention that fact. We subjoin the first two steps of an example from page 156 of the *Praxis*, which is quoted in full by De Morgan.

[6] Vieta, p. 228.

[7] The writer has been unable to consult the original and depends for his description on the following secondary sources: De Morgan's article, "Notices on the Progress of the Problem of Evolution," in *Companion to the British Almanac, 1839*, pp. 34–52; Hutton's *Tracts*, II, pp. 278–86; Cantor, II (1900); Henry Stevens, *Thomas Hariot and his Associates*, London, 1900; J. Wallis, *A Treatise of Algebra both Historical and Practical*, London, 1685.

"Sit aequatio numerosè proposita
$aaaa - 1024aa + 6254a = 19633735875$
Radix universalis successivè educenda.

		3 7 5
Homogeneum resolvendum	$hhhh$	19633735875
	ggg	6254
	$-ff$	$-$ 1024
	bbb	$-$ 27
	$+$	27006254
	Divisor A	269003854
	$gggb$	18762
Radix sing. prima $b = 3$,	$-ffbb$	$-$ 9216
	$bbbb$	81
	$+$	81018762
	Ablatitium Ab	80097162
	Radix singularis	3
Homogeneum residuum resolvendum		11624019675
	ggg	6254
	$-ff$	$-$ 1024
	$-2ffb$	$-$61440
Radix sing. decupl. $b = 30$,	$4bbb$	108000
	$6bb$	5400
	$4b$	120
	$+$	113526254
	$-$	62464
	Divisor B	112901614
	$gggc$	43778
	$-ffcc$	$-$ 50176
	$-2ffbc$	$-$ 430080
Radix sing. secund. $c = 7$	$4bbcc$	756000
	$6bbcc$	264600
	$4bccc$	41160
	$cccc$	2401
	$+$	1064204778
	$-$	480256
	Ablatitium Bc	1059402218"

Harriot's canonical form is $aaaa - ffaa + ggga = hhhh$. He lets $a = b + c$. Then the transformed equation becomes $(bbbb + 4bbbc + 6bbcc + 4bccc + cccc) - ff\,(bb + 2bc + cc) + ggg\,(b + c) = hhhh$. By Vieta's method the divisor would be $(ggg - ff - 2ffb + 4bbb + 6bb + 4b)$; and this is identically the divisor in the illustration quoted, both in the second and the third step.

Harriot's *Praxis* was the most analytical algebra that had been written up to that time. His lucid treatment of the theory of equations, his improved symbolism, and the inspiration of his personality to his students were accountable in no small measure for the popularity of Vieta's method in England, and it was often called after him "Harriot's method."

3. THE IMPROVEMENTS OF OUGHTRED AND WALLIS

No one did more to popularize the new method than did the clergyman mathematician William Oughtred. This he accomplished by giving private tuition to ambitious young men and these spread his teachings throughout Great Britain; among them were Seth Ward, Christopher Wren, and John Wallis. An even more potent influence was his textbook, the *Clavis Mathematica*. Before the year 1700 five editions of this little volume had gone through the press.

The first edition appeared in 1631, the same year in which Harriot's *Praxis* was published. The *Clavis* appeared first. In it there is no reference to Harriot, nor any use made of his improvements: the two men evidently knew nothing of one another's work. Oughtred uses Vieta's method throughout. In "locating" the root he depresses the equation and uses a method similar to that of Stevin for obtaining the first digits of the root. It is interesting to observe that he uses the decimal bar and logarithms in his solutions.

John Wallis made this method more practical by arranging the work in a compact, yet clear form, so as to show the steps in the computation. He differed from Vieta, Harriot, and Oughtred in not rejecting the coefficient unity in his divisor. To help discover cases of devolution and anticipation he employs two sets of periods, one for the resolvend and another for the coefficient; at times he uses Stevin's rule for this purpose.

4. THE CONTRIBUTION OF HUME

In France the method of Vieta made slower progress. In 1636, James Hume, an emigrated Scotchman, published in Paris a work with the title *Algebre de Viete, d'une methode nouvelle, claire, et facile.* It is an abridged presentation of Vieta's analytical works, reproducing both his methods and his illustrations. However, he introduces one modification: he uses only so much of the divisor and of the dividend as is needed to compute the next digit. Thus in the example (2) which we quoted from Vieta he uses 12 for the divisor and 6350 for the dividend, where Vieta uses 126,030 and 6,350,197.[8]

Vieta's method was adopted by P. Hérigone[9] in his textbook *Cursus Mathematicus*, published about this time; but neither Hume's nor Harriot's improvements were used.

5. SUMMARY

1. The Hindu method of finding the roots of large numbers was extended by Vieta to the solution of complete equations in his *De Numerosa Potestatum* (1600).

2. The method was first taken up in England by Harriot (some time before 1621), who modified it by using canonical equations in which the second term had been eliminated.

3. Oughtred (1631) and Wallis (1685) improved the arrangement, but otherwise used Vieta's method unmodified, except that Wallis included the coefficient unity in his divisor.

4. In France James Hume (1636) used only so much of the dividend and divisor as was necessary to evolve the next digit of the root. Hérigone's algebra (1642) was the first French textbook to popularize Vieta's method in France.

[8] Hume, *Algebre de Viete*, pp. 234–38.

[9] Hérigone, *Cursus Mathematicus*, II, Paris, 1642, pp. 270–76.

VII

NEWTON'S METHOD OF APPROXIMATION

1. ADVANCE IN THE THEORY OF EQUATIONS

The seventeenth century saw a remarkable advance in the theory of equations. The symbolic notation introduced by Vieta and improved by succeeding algebraists, notably by Harriot and Descartes, made the symbolic equation a chief instrument of analysis. When Harriot put the equation in the form $a_0x^{n-1} + a_1x^{n-1} + \ldots\ldots + K = 0$, the eye began to aid the mind in grasping the relations of the parts of an equation. The law of signs, used by Harriot, enunciated by Descartes, and refined by Newton, gave an effective, though limited, criterion for the number and nature of the roots in an equation. The invention of coordinate geometry by Descartes in 1637 and the independent discoveries of the differential calculus by Newton and Leibniz a few years later gave a powerful impetus to work in theory of equations and in practical numerical calculation.

For purposes of numerical approximation one of the tools made most effective by the improved symbolism was the infinite series in an equation. Series, as a means of exhaustion and of summation, had been successfully used for approximating numerical values since the time of Archimedes. But with the introduction of symbolic algebra, series gained an effectiveness undreamed of by the ancients. Pioneers in the use of equations in series for purposes of numerical approximation were Wallis (1656), Lord Brouncker (1620–1684), Gregory (1667), and Newton (1669).

2. A DESCRIPTION OF NEWTON'S PROCESS

It was while finding the area under curves by his newly-invented "method of fluxions" that Isaac Newton (1642–1727) discovered a new and most necessary application of series. Following the lead of Wallis he transformed such fractional and radical expressions as $y = \dfrac{a^2}{b + x}$ and $y = \sqrt{a^2 + x^2}$ into integrable form by expanding them into equations in series by ordinary division and finding of

roots. This scheme he extended to affected equations. "Since the whole difficulty lies in the solution," he says, in explaining it, "I shall first illustrate the method I use, in a numerical equation." [1]

Such then is the historical setting of the beginnings of a type of approximation by successive substitutions that is almost universally known as "Newton's method of approximation." It was entirely incidental to his work in quadrature by calculus. The originator of this famous method gives only a single numerical example, and it is illustrative and introductory to the more general problem of deriving a method for finding the root of literal equations by means of equations in series. Some historians notwithstanding, it is hardly correct to say that it was a further development or an adaptation of Vieta's method. Of course, Newton's method includes Vieta's ideas of limit, substitution, and exhaustion; but these seem inherent to any method not using double false position. In some of its features Newton's solution resembles the approximation processes in Chelebî's annotations and of Stevin. Later algebraists[2] sometimes used "Stevin's rule" of substituting 1,10,100 etc., to find the integral part of the root. Newton himself says of this method: "Whether this method of solving equations be commonly practised or not, I cannot tell; but surely to me it appears simple in comparison with other methods and more suited to practise."[3]

Newton's solitary numerical example, the solution of $y^3 - 2y - 5 = 0$, has become classic. Lagrange, Ruffini, and Horner, all demonstrated the virtue of their methods by applying it to Newton's equation, and it has become a favorite problem in text-books.

This process of approximating by successive substitution in equations is thus explained by Newton:

Suppose that an approximate value $y = 2$ has already been found, differing by not more than one tenth of itself from the true root. Then $y = 2 + p$. Substituting this value in the original equation gives $p^3 + 6p^2 + 10p - 1 = 0$. Since p is a fraction we can reject all but the linear terms, or $10p - 1 = 0$ approximately. That is $p = 0.1$, approximately; or $p = 0.1 + q$. Continuing this process as far as desired we get several values q, r, s, etc. And

[1] Newton, *Analysis by Equations of an infinite Number of Terms*, translated and edited with commentaries, by John Stewart, London, 1745, pp. 321–28, 357.

[2] N. Saunderson, *Elements of Algebra*, II, Cambridge, 1790, p. 728.

[3] Newton, *Analysis by Equations*, p. 330.

$y = p + q + r + s + \ldots\ldots$ If at any time the coefficients are such that one doubts the accuracy given by the linear terms, one may advantageously use the quadratic part of the equation. Thus $6p^2 + 10p - 1 = 0$ would give a more accurate value of p than $10p - 1 = 0$. In such cases, use the lesser root of the quadratic. Newton sets up his solution in paradigm form, as follows:[4]

$y^3-2y-5=0$		$+2{,}10000000$
		$-0{,}00544853$
		$+2{,}09455147=y$
$2+p=y$	$+y^3$	$+8+12p+6p^2+p^3$
	$+2y$	$-4\ -2p$
	-5	-5
	sum	$-1+10p+6p^2+p^3$
$0,\ 1+q=p$	$+p^3$	$+0{,}001+0{,}03q+0{,}3q^2+q^3$
	$+6p^2$	$+0{,}06\ +1{,}2\quad +6{,}0$
	$+10p$	$+1,\quad +10$
	-1	$-1,$
	sum	$+0{,}061+11{,}23q+6{,}3q^2+q^3$
$-0{,}0054+r=q$	$+6{,}3q^2$	$+0{,}000183708-0{,}06804r+6{,}3r^2$
	$+11{,}23q$	$-0{,}060642\quad +11{,}23$
	$+0{,}061$	$+0{,}061$
	sum	$+0{,}000541708+11{,}16196r+6{,}3r^2$
$-0{,}00004854+s=r$		

He next takes up the more general method of literal equations.[5] "These things being thus shown in numbers, let $y^3 + a^2y - 2a^3 + axy - x^3 = 0$ be proposed to be solved." Using the same method, setting $y = a + p$, he derives $p = -\frac{x}{4} + q$; $q = \frac{x^2}{64a} + r$, etc., and $y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{1638a^3} + \ldots\ldots$ A practical difficulty

[4] Newton, *Analysis by Equations*, pp. 328–329.
[5] Newton, *Analysis by Equations*, pp. 330–343.

in getting the supplementary approximation is encountered when the coefficients are letters. He reduces this by arranging the terms according to their dimensionality; this arrangement has been called Newton's Parallelogram. The value of y will converge when coefficients are so chosen that x is small; and the smaller they are, the more rapidly the series converges. Later algebraists sometimes found numerical approximations by using as forms such solutions of literal equations. But Newton, as might be expected, did not substitute numerical values of x until after the integration. With him the objective was the area of the curve and not the numerical value of y.

Newton's description of his method was contained in an unpublished tract entitled *De Analysi per aequationis numero terminorum infinitas* which was shown to Barrow and Collins in 1669.[6] His method was first given to the public in the 1685 and 1693 editions of Wallis's algebra. In 1704 and 1711 it was published in Newton's own works in Latin and in 1736 and 1745 it appeared in Colson's and Stewart's translations. Newton's name was always linked with his particular procedure, but the appelation "Newton's method" came to be applied to this whole type of approximations through the writings of Lagrange and Fourier.

3. RAPHSON'S DEVELOPMENT OF NEWTON'S METHOD

The first mathematician to enlarge upon the new method of approximating roots by substitution in equations of infinite series was Joseph Raphson. We may even say that he was the first to expand it into a system. It was he who made Newton's method a component part of modern algebra; for with Newton it was an accessory to the calculus, and was never taken up in his *Universal Arithmetic*. In 1690 Raphson published a tract of forty-eight pages entitled *Analysis Aequationum Universalis seu Ad Aequationes Algebraicas Resolvendas Methodus Generalis, & Expedita, Ex nova Infinitarum Serierum Methodo.* In 1697 he published a second edition with an appendix discussing the recently published methods of DeLagny and Halley. Though quoting from Wallis [7] as to

[6] Letter to Mr. Oldenburg, Oct. 24, 1676, given in Stewart's commentary on *Analysis by Equations*, p. 358.

[7] Raphson, *Analysis Aequationum Universalis*, London, 1697, p. 50.

Newton's work in infinite series, he does not even refer to Newton's method of approximation.

Raphson's method is eclectic. He sets up the example like Vieta, employs Harriot's notation, and divides the coefficients and resolvends into periods, "modo, quem ducere *Vieta*, nostratisque *Harriotus* & *Oughtredus*." Like Oughtred he suggests the use of logarithms in finding the first digits of the root; he adds that it may also be done by constructing parabolas and by examining the limits of the roots. But the main contributing elements in his method are Harriot's use of canonical forms and Newton's method of substitution. Employing and enlarging upon these ideas Raphson builds a canonical form for the supplement of the root in each of sixty-four canonical equations; so that instead of substituting anew in each equation, as in Newton's example, he could substitute the successive approximations in one canonical form. The "Newtonian divisor" in these forms is a misnomer and should be credited to Raphson. The name given to the collection of forms, "Canones Directorii," is the same as that used by Harriot.

Only the two equations $a^3 = d$ and $ba - a^3 = c$, representing pure and affected powers respectively, have the canonical forms for their supplements derived in full. Their derivations are as follows: [8]

(1) If $a^3 = d$, and $a = g + x$, then we have $g^3 + 3g^2x + 3gx^2 + x^3 = d$. Rejecting the powers of x above the first, as being comparatively small, we have the supplement $x = \dfrac{d - g^3}{3g^2}$.

(2) If $ba - a^3 = c$, and $a = g + x$, then "e Theoremata Vietae," $bg - g^3 + (b - 3g^2)x - 3gx^2 - x^3 = c$. Again, rejecting the powers of x above the first, we have for the supplement $x = \dfrac{c + g^3 - bg.}{b - 3g^2}$

Then he proceeds to prove the convergency of the series.

We append the following problem [9] solved by this last form:

"Numeris $\dot{7}7\dot{2}8\dot{4}\,a - aaa = \dot{8}01\ \dot{3}12\ \dot{8}$

[8] Raphson, pp. 5–6, 8.

[9] Raphson, p. 19.

$$\text{Theor. } x = \frac{c + ggg - bg}{b - 3gg}$$

$$\begin{array}{lll}
 & g = 200 & \\
c + ggg - bg = & +626328 & \\
+ b - 3gg = & -42716) + 626328(-14 = x & \\
\hline
 & 200 & \\
 & \underline{-14} & \\
 & g = 186 & \\
c + ggg - bg = & +143\ 160 & \\
+ b - 3gg = & -26504) + 143\ 160(-5.4 = x & \\
\hline
 & 186 & \\
 & \underline{-5.4} & \\
 & g = 180.6 & \\
c + ggg - bg = & +16152.216 & \\
+ b - 3gg = & -20565.08) + 16152.216(-.7854 = x & \\
\hline
 & 180.6 & \\
 & \underline{-.7845} & \\
 & g = 179.8146 \text{ vel } 179.814 = g & \\
c + ggg - bg = & +322.299\ 405\ 144 & \\
+ b - 3gg = & -19715.224) + 322.299405144(-.016348 = x & \\
\hline
 & 179.814 & \\
 & \underline{-.016\ 348} & \\
 & a = 179.797652.\text{''} &
\end{array}$$

The tract contains the solutions of thirty-four numerical problems taken mostly from geometry and trigonometry. One problem, indeed, deals with interest and is undoubtedly the first instance of Newton's method being used in the solution of exponential equations.[10] Using $x = \dfrac{b - g^{365}}{365\, g^{364}}$ as the canon form for the supplement of the equation $a^{365} = b$, he solves quite accurately the equation $a^{365} = \$1.06$.

4. THE APPROXIMATIONS OF DELAGNY

Thomas Fantet DeLagny (1660–1734) spent over forty years in work on the approximation of roots. Observing that the expression

[10] Raphson, p. 36, problem 29.

$z = \sqrt{a^2 + b}$ has a rational approximation $z = a + \frac{b}{2a}$ and a radical approximation $z = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b}$, he sought to obtain similar expressions for $z = \sqrt[3]{a^3 + b}$. These he found to be $\frac{a}{2} + \sqrt{\frac{a^2}{4} + \frac{b}{3a}} >$ $z > \frac{a}{2} + \sqrt{\frac{a^2}{4} + \frac{b-1}{3a}}$ and $a + \frac{ab + a}{3a^2 + b + 1} > z > a + \frac{ab}{3a^2 + b}$.[11] These results were published in 1691 [12] and in 1697 he incorporated them in the third part of his *Nouveaux Élémens d'Arithmétique et d'Algèbre*, and later in his *Analyse Générale* (1733). He derived similar expressions for higher powers. The process itself we shall discuss in connection with Halley's general formula. But the reader will be interested to know that in finding the roots of large numbers, DeLagny preferred to use the formula $\sqrt[3]{a^3 + b} = a + \frac{ab}{3a^2 + b}$ rather than the Hindu method. Thus he finds the cube root of 817,400,375 as follows: $\sqrt[3]{817} = \sqrt[3]{729 + 88} = 9\,\frac{792}{2275}$; multiply this by 100, since the desired root has three digits; the result is 935.

DeLagny's method of attack in solving both pure and affected equations may be seen from his solution of the equation $x^3 = a^2x + b$.[13] Let $x = a + y$. Then $x^3 = a^3 + 3a^2y + 3ay^2 + y^3$; also $x^3 = a^3 + a^2y + b$. Therefore $b = 3ay^2 + 2a^2y + y^3$. Neglecting y^3 as being very small, $3ay^2 + 2a^2y - b = 0$, or $y = -\frac{a}{3} + \sqrt{\frac{a}{9} + \frac{b}{3a}}$. Consequently, $x = \frac{2a}{3} + \sqrt{\frac{a^2}{9} + \frac{b}{3a}}$. Replacing b by $b - y^3$ in the

[11] The incorrect data given in Cantor, III (1898), p. 115, seem to be taken from Halley's article in the *Philosophical Transactions* for 1694.

[12] DeLagny, *Nouveaux Élémens*, Paris, 1697, pp. 303–6. *Analyse Générale*, Paris, 1733, pp. 336, 342–46, 362–94. *Journal des Sçavans*, May 14, 1691, pp. 297–98.

[13] DeLagny, *Nouveaux Élémens*, pp. 473–74.

last expression gives $x = \frac{2a}{3} + \sqrt{\frac{a^2}{9} + \frac{b - y^3}{3a}}$. The formulas for x and y are canonical expressions for the successive roots and supplements of the equation $x^3 = a^2x + b$. Raphson's canonical form in the equivalent equation $a^3 - ba = c$ is $a = g + \frac{c + bg + g^3}{3g^2 - b}$, different in form and derived by a different process. In DeLagny's illustration, $x^3 = 7569x + 240{,}903$; then $a^2 = 7569$, $a = 87$, $b = 240{,}903$; this gives $x = 58 + \sqrt{1764} = 100$ for a first approximation, $x = 58 + \sqrt{1755\frac{152}{261}}$ for a second approximation, and so on.

DeLagny's "Method of Mediation" resembled the more effective Rule of Mean Numbers given by Chuquet; except that DeLagny used the arithmetical mean.[14]

Two points about DeLagny's work are worthy of special mention. He stood alone among his contemporaries in his insistence on a system of gauging the error of the approximation. "Every method is useless," he says, "if it be not accompanied by another method which gives the limits of error, be it *par excès* or *par défaut*." [15] And it was his pride that in the *Triangle du Rapport* he had an unfailing scheme for detecting the limits of error.

The other point relates to his ambitious scheme of facilitating the solution of higher degree equations by means of tables. For this purpose he worked out three elaborate sets: (1) tables for solving numerical equations directly, covering forty-nine large pages; (2) the "Trapese Logarithmique," [16] for facilitating by means of arithmetical progression the solution of equations; (3) the "Triangle du Rapport" for expanding the roots into series. These tables together with Raphson's Canones Directorii might have come to fill a practical need, as do our integral and logarithm tables of to-day, had they not been made obsolete by the discoveries of Halley and Taylor.

DeLagny's style was burdened with much verbiage and his classifications and arrangement lacked sharp lines. That may

[14] *Nouveaux Élémens*, pp. 507–11.

[15] *Analyse Générale*, p. 546.

[16] *Analyse Générale*, pp. 320–323; see also *Mémoires de l'Académie* for 1705 and 1706.

account somewhat for the comparatively slight influence of his voluminous work, some parts of which are very scholarly. Also, if he had lived in England instead of in France, his influence might have been more widespread. At any rate, the forty-eight pages of Raphson left a greater immediate impress on approximation methods than did the twelve hundred pages of DeLagny.

5. HALLEY'S INVENTION OF GENERAL FORMULAS

Dr. Edmund Halley (1656–1742), the English astronomer, had learned, through a friend, of DeLagny's results for $\sqrt[3]{a^3+b}$ and $\sqrt[5]{a^5+b}$, and this inspired him to investigate approximation methods for such expressions. It is amusing to know that a garbled version of these results had reached Halley. For the cubic, for instance, DeLagny found that $\frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{b-1}{3a}} < \sqrt[3]{a^3+b} < \frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{b}{3a}}$; also $a+\frac{ab}{3a^3+b} < \sqrt[3]{a^3+b} < a+\frac{ab+a}{3a^3+b+1}$. It reached Halley in the form $a+\frac{ab}{3a^3+b} < \sqrt{a^3+b} < \frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{b}{3a}}$. As no demonstration accompanied the report of DeLagny's work, he derived these statements by a method of his own; this was analogous to that of the French scholar and became the basis of his later investigations.[17] He published his results in the *Philosophical Transactions* for 1694 in an article entitled "A new exact, and easy method of finding the roots of any equations generally, and that without any previous reduction." In this article he differs from Raphson and Halley in that he gives frank and generous recognition to the work of his contemporaries.

His approximation for $\sqrt[3]{a^3+b}$ is as follows:

(1) Suppose a^3+b to be a non-cubic number; also suppose that $a^3+b=(a+e)^3$, approximately, so that $a^3<(a+e)^3<(a+1)^3$.

Then $(a+e)^3 = a^3+3a^2e+3ae^2+e^3 = a^3+b$ and $(a+1)^3 = a^3+3a^2+3a+1$; hence $e^3<1$.

[17] *Philosophical Transaction of the Royal Society of London*, No. 210 (1694), p. 136; abridged English translation by Hutton, Shaw, and Pearson, III (1683–1694), pp. 640–49. Hereafter referred to as *Phil. Trans.* (Abr.).

Also $b = 3a^2e + 3ae^2 + e^3$; or $b = 3a^2e + 3ae^2$, approximately. Since a^2e is so much greater than ae^2, he thinks of e as equal to $\frac{b}{3a^2}$.

From $b = 3a^2e + 3ae^2$, he gets $e = \dfrac{b}{3a^3 + 3ae} = \dfrac{b}{3a^3 + 3a\left(\dfrac{b}{3a^2}\right)}$ $= \dfrac{ab}{3a^3 + b}$.

Therefore $\sqrt{a^3 + b} = a + e = a + \dfrac{ab}{3a^3 + b}$. This is the rational form, which by his handling of e is slightly less than $a + e$.

(2) Since $b = 3a^2e + 3ae^2$, it follows that $e^2 + ae = \dfrac{b}{3a}$; completing the square and finding roots gives the radical form $a + e = \dfrac{a}{2} + \sqrt{\dfrac{a^2}{4} + \dfrac{b}{3a}}$, which is a little too large. Similar processes give $a - \dfrac{ab}{3a^3 - b} < \sqrt{a^3 - b} < \dfrac{a}{2} + \sqrt{\dfrac{a^2}{4} - \dfrac{b}{3a}}$. The other results reported by DeLagny are then verified.

So far Halley has only re-discovered DeLagny's results. But now he takes the important step which constitutes his own real contribution to facilitating the operation of Newton's method. First he derives a general form for all pure powers; thereupon he extends the method of computation as well as its generalization to affected equations. It was this important feature that led to Taylor's important discovery. Halley says: "While I was considering this matter, I fell upon a general method of forms for any power whatever, which is elegant enough, and which I cannot prevail on myself to conceal, since in the higher powers they at least triple the known figures of the root. And these forms, both rational and irrational, proceed as follows:

$$\sqrt[3]{a^2+b} = \frac{0}{1}a + \sqrt{\frac{1}{1}aa + \frac{b}{1a^0}}, \text{ or } a + \frac{ab}{2a^2 + \frac{1}{2}b}.$$

$$\sqrt[3]{a^3+b} = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{b}{3a}}, \text{ or } a + \frac{ab}{3a^3 + \frac{2}{2}b}.$$

$$\sqrt[4]{a^4+b} = \frac{2}{3}a + \sqrt{\frac{1}{9}aa + \frac{b}{6a^2}}, \text{ or } a + \frac{ab}{4a^4 + \frac{3}{2}b}.$$

$$\sqrt[5]{a^5+b} = \frac{3}{4}a + \sqrt{\frac{1}{16}aa + \frac{b}{10a^3}}, \text{ or } a + \frac{ab}{5a^5 + \frac{4}{2}b}.$$

$$\sqrt[6]{a^6+b} = \frac{4}{5}a + \sqrt{\frac{1}{25}aa + \frac{b}{15a^4}}, \text{ or } a + \frac{ab}{6a^6 + \frac{5}{2}b}.$$

And so on for the higher powers."

For the purpose of extending this method to affected equations of any degree he builds up what he calls a "general analytical speculum." This is a table of powers to aid in the transforming of the equation $cz + dz^2 + fz^3 + \ldots\ldots = 0$ by letting $z = a + e$. The resulting coefficients of e, e^2, e^3, etc., are represented by s, t, u, etc. Following is his

Table of Powers

			s	t	u	w	x	y
cz	$=$	ca	$+ ce$					
dz^2	$=$	da^2	$+ 2dae$	$+ de^2$				
fz^3	$=$	fa^3	$+ 3fa^2e$	$+ 3fae^2$	$+ ffe^3$			
gz^4	$=$	ga^4	$+ 4ga^3e$	$+ 6ga^2e^2$	$+ 4gae^3$	$+ ge^4$		
hz^5	$=$	ha^5	$+ 5ha^4e$	$+ 10ha^3e^2$	$+ 10ha^2e^3$	$+ 5hae^4$	$+ he^5$	
kz^6	$=$	ka^6	$+ 6ka^5e$	$+ 15ka^4e^2$	$+ 20ka^3e^3$	$+ 15ka^2e^4$	$+ 5kae^5$	$+ ke^6$

The modern reader recognizes the sums of the coefficients in the different columns to be the same as in the expansion $f(z) = f(a+e)$ $= f(a) + f'(a)e + \frac{f''(a)}{2!}e^2 + \frac{f'''(a)}{3!}e^3 + \ldots$ But Halley denotes the

result in the form analogous to that which he uses for pure powers, viz., $b = se + te^2 + ue^3 + we^4 + xe^5 + ye^6 +$, where $b = f(z) - f(a)$.

Operating with this equation he finds that the approximate roots of z are: $z = a \pm \dfrac{sb}{s^2 \pm tb}$, in the rational form, and $z = a \mp$

$$\frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}s^2 \mp bt}}{t}$$ in the radical form.

Among his numerical illustrations is the equation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$,[18] taken from Wallis. Raphson uses the same illustration. Even a cursory examination shows the merits of Halley's General Analytical Speculum. The arrangement is still used in some modern texts on the theory of equations.

6. TAYLOR'S APPLICATION OF THE CALCULUS TO HALLEY'S COEFFICIENTS

The next improvement came from the field of the differential calculus. Brooks Taylor (1685–1731) had shown in his *Methodus Incrementorum* that

$$f(a + e) = f(a) + f'(a)e + \frac{f''(a)}{2!}e^2 + \frac{f'''(a)}{3!}e^3 + \ldots\ldots$$ (Taylor's Series).

In an article in the *Philosophical Transactions* for 1717[19] entitled "An attempt towards the improvement of the method of approximating, in the extraction of the roots of equations in numbers," he shows the identity of the coefficients in his series and the coefficients in Halley's General Analytical Speculum. Since his series is general, applicable to polynomials as well as binomials, to fractional as well as integral exponents, to transcendental as well as algebraic expressions, he observes that Halley's work can easily be extended into these fields by aid of the said series. He illustrates this by solving the equations $(y^2 + 1)^{\frac{1}{2}} + y - 16 = 0$ and $\log y = 0.29$.

[18] *Phil. Trans.* (Abr.), p. 648; Wallis, *Algebra*, pp. 233–35; Raphson, pp. 25–26.

[19] *Phil. Trans.* (Abr.), VI (1713–1723) pp. 299–304; *Phil. Trans.*, Vol. 30, pp. 610–22.

7. FURTHER SIMPLIFICATIONS OF NEWTON'S METHOD

In modern text-books we find the Newtonian method of approximation stated thus: if $f(x) = 0$, and a is a first approximation, then $x = a - \dfrac{f(a)}{f'(a)}$ is a second approximation. Newton and Raphson had used these values, but obtained them by transforming the equation. Halley used the quadratic part of the transformed equation; Newton did the same at times. Brooks Taylor also used the quadratic, but obtained his coefficient by differentiation. The first writer to use the linear differential formula $x = a - \dfrac{f(a)}{f'(a)}$ was Thomas Simpson in his essay, "A new Method for the Solution of Equations in Numbers" (1740).[20] He states the formula rhetorically and illustrates it by this example: $f(x) = 300x - x^3 - 1000 = 0$; $x = a + A$; by trial a is found to lie between 3 and 4; try $a = 3.5$; $f'(x) = 300 - 3x^2$; $f(a) \div f'(a) = A$; $f(3.5) = 7.125$; $f'(3.5) = 263.25$; $A = 0.027$; $3.5 - 0.027 = 3.473$.

Proceeding as before, letting $a = 3.473$, we have $x = 3.47296351$.

Simpson uses Newton's method in solving simultaneous equations. His fifth example is a set of exponential equations: $x^x + y^y - 1000 = 0$ and $x^y + y^x - 100 = 0$.

De Courtivron (1744), Euler (1755), and Lambert (*c.* 1770) also applied the calculus to the solution of higher equations.[21] In his algebra [22] (1770), Euler discards the higher powers in both dividend and divisor during the substitution instead of carrying them along until the division takes place. Thus in solving the equation $x^3 + ax^2 + bx + c = 0$, he sets $x = n - p$, $x^2 = n^2 - 2np$, $x^3 = n^3 - 3n^2p$; from these four expressions he finds $p = \dfrac{n^3 + an^2 + bn + c}{3n^2 + 2an + b}$. Edward Waring (1770) extended the Newtonian approximation process into the realm of complex roots. The functional expression $x = a - \dfrac{f(a)}{f'(a)}$ is due to Lagrange [23] (1798).

[20] Simpson, *Essays on Mathematics*, London, 1740, p. 81.

[21] Euler, *Institutiones calculi differentialis*, sections 224, 234, 235; *Histoire de l'Académie*, 1744, p. 405–14.

[22] Euler, *Algebra*, I (1770), Chap. 16.

[23] Lagrange, *Traité de la Résolution des Équations Numériques* (1798), Note XI.

8. SUMMARY

1. The method of solving equations of higher degree by successive substitutions in derived equations was originated by Newton (1669) incidental to his getting integral expressions for his work in areas. He performed a transformation for every new supplement to the root. His method was first made public in the algebra of Wallis (1685).

2. Raphson (1690) worked Newton's method into a system. Following the example of Harriot he derived canonical forms to simplify calculation. In his plan only one transformation and one divisor are needed for each equation.

3. DeLagny (1691) derived notable approximations in the solution of pure equations by a special adaptation of Newton's method, and worked out an elaborate set of tables for the solution of numerical equations.

4. Halley (1694) generalized DeLagny's method for pure equations and extended it to affected equations; in his table of powers he gave a general formula for the coefficients of transformed equations. This made obsolete Raphson's canonical forms and DeLagny's tables.

5. Taylor (1717) discovered that Halley's coefficients were the same as the coefficients in Taylor's series. This made possible the direct use of the calculus in finding roots.

6. Our present standard linear approximation formula $x = a - \frac{f(a)}{f'(a)}$ was first used by Simpson (1740); it received its conventional form through the writings of Lagrange (1798).

VIII

CERTAIN EFFECTIVE BUT NON-PRACTICABLE METHODS

1. THE ASCENDANCY OF NEWTON'S METHOD

Newton's method of approximating roots by substituting values in equations of an infinite number of terms did not immediately upon its publication in 1685 supersede Vieta's method of finding roots by evolving the successive digits. The latter was still used by Dechâles in 1690 and appears as late as 1702 in a new edition of Oughtred's *Clavis*. It seems as if the transition from the one to the other must have been facilitated by the eclectic character of Raphson's arrangement. Halley's general formula increased the popularity of the new method. Its ascendancy is clearly shown in an article written by Wallis in 1694. After Taylor's discovery (1717) it had no serious rival old or new, until the publication of Horner's method in 1819. But new methods sprang up to challenge it in this period, and we shall describe the three that gave the most promise.

2. ROLLE'S METHOD OF CASCADES

A considerable contribution to the location of roots was made by Michel Rolle in his two works *Traité d'Algèbre* (1690) and *Démonstratio d'une Méthode pour Résoudre les Égalitez de toutes Degrez* (1691). He demonstrated that there cannot be more than one real root of $f(x) = 0$ between two successive real roots of $f'(x) = 0$ (Rolle's Theorem). He invented a method of finding roots, called the "Method of Cascades"; it consisted in locating the roots by first finding the limits ("hypotheses") of the roots of the different cascades. Between the roots of each cascade lie the roots of the next higher cascade, and so on.[1] It was an anticipation of the method of using successive derivatives (Rolle's cascades) now used in calculus for locating roots.

With Rolle an equation is "prepared" when it has the form $x^n - a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \ldots\ldots = 0$ where the a's are posi-

[1] *Traité d'Algèbre*, p. 124, p. 103; p. 118; *Démonstratio*, article VI (New York Public Library); Cajori in *Bibl. Math.* XI (1910–11), p. 300–313.

tive integers. Then the upper limit of the root is found by dividing the largest negative coefficient by the highest exponent, and adding unity and as much more as necessary to make the result an integer. His Method of Cascades he illustrates thus:[2]

Let $x^3 - 57x^2 + 936x - 3780 = 0$ (third cascade).

Multiply the terms of the cascade by parts of the progression 3, 2, 1, 0, respectively; divide by x. Then $3x^2 - 114x + 936 = 0$ (second cascade).

Multiply by 2, 1, 0; divide by $2x$. Then $3x - 57 = 0$ (first cascade).

The "hypotheses" for the second cascade are 0, 19 and 39 (or $\frac{114}{3} + 1$).

The "hypotheses" for the first cascade are 0 and 19.

Between 0 and 19, the second cascade is found to have 12 as a root; and another root, 21, between 19 and 39.

Finally for the third cascade the extreme "hypotheses" are 0 and 3781 $\left(\text{or } \frac{3780}{1} + 1\right)$; 0, 12, 26, and 3781 are its four "hypotheses."

The roots of the third cascade (that is, the given equation) can then be found to be: 6 between 0 and 12; 21 between 12 and 26; 30 between 26 and 3781.

3. THE METHOD OF RECURRING SERIES

De Moivre (1667–1754) first conceived of series in which each coefficient bears a given relation to certain of the preceding coefficients; such series he called recurring series (1720).[3] Daniel Bernoulli (1700–1782) first used them for approximating roots (1728). Following is his process:[4]

Take such an equation as $1 = ax + bx^2 + cx^3 + dx^4$. Select arbitrarily the four terms A, B, C, D. Then E, F, G, H, etc., constitute a recurring series if

[2] *Traité d'Algèbre*, pp. 127–28.

[3] *Phil. Trans.*, Vol. 32 (1722), pp. 162–78; DeMoivre, *Miscellanea Analytica*, London, 1730, Bk. I, II.

[4] Cantor, III (1898), pp. 621–22.

$$E = aD + bC + cB + dA,$$
$$F = aE + bD + cC + dB,$$
$$G = aF + bE + cD + dC,$$
$$H = aG + bF + cE + aD;$$

and the approximate values of x are $\frac{E}{F}, \frac{F}{G}$, etc.

In solving the equation $1 = -2x+5x^2 - 4x^3 + x^4$, Bernoulli sets $A = B = C = D = 1$, and obtains the series

$0, 2, -7, 25, -93, 341, -1254.$

This gives $\frac{0}{2}, -\frac{2}{7}, -\frac{7}{25}, \ldots\ldots \frac{341}{1254}$ as the increasingly close approximations of the root. The last value, when substituted in the given equation gives $1 = 0.999487$, a very small error.

Bernoulli gave no analytical proof for the validity of his process. That was supplied by Euler (1748) and later by Lagrange (1798).[5]

A considerable improvement was made by Euler in his algebra (1770). He shows the general relation that must exist for the different powers, and thus immediately gets a recursion formula for any equation under consideration.[6] The recursion formula performs a function similar to that of Raphson's canonical form or of Halley's general formula.

Suppose our series p, q, r, s, t, etc., is such that $\frac{q}{p}, \frac{r}{q}$, etc., express approximate values of x. Then: $\frac{q}{p} = x; \frac{r}{q} = x$; multiplying, $\frac{r}{p} = x^2$; also, $\frac{s}{r} = x$; hence $\frac{s}{p} = x^3$; similarly $\frac{t}{p} = x^4$, etc. Not only do we have a substitution table for powers, but we have one with the first term as the common denominator.

As an illustration, let us substitute these values in the equation $x^3 = x^2 + 2x + 1$. This gives $s = r + 2q + p$. Using this as a re-

[5] Euler's *Introductio in Analysin Infinitorum*, Chap. 17; Lagrange, *Traité de la Résolution des Équations Numériques*, Note VI.

[6] Euler's *Algebra*, Vol. I, Chap. 16.

cursion formula for s in terms of r, q, and p, we can arbitrarily select these to be 0, 0, 1. We then get as our series 0, 0, 1, 1, 3, 6, 13, 28 60, 129, etc. Testing for $x = \frac{60}{28}$ in the original equation we get an error of $\frac{13}{343}$. A closer result would be given by $x = \frac{129}{60}$.

Not all equations admit of the application of the method of recurring series. If no common limit results, the equation must first be transformed. This happens particularly if the second term is lacking. Thus foı the equation $x^2 = 2$ we get the recurring series 1, 1, 2, 2, 4, 4, 8, 8, etc. A similar situation arises for $x^3 = 2$. To obviate this, we let $x = y - 1$. In the resulting equation $y^3 = 3y^2 - 3y + 1$ we get the recursion formula $s = 3r - 3q + p$, and this gives us the series 0, 0, 1, 3, 6, 12, 27, 63, 144, 324. This gives us $y = \frac{324}{144}$ and $x = \frac{5}{4}$. Checking will reveal an error of $\frac{3}{64}$.

4. THE METHOD OF CONTINUED FRACTIONS

Newton's method had one serious defect: it was inoperative if two roots were close together, for then the series would not converge, and might after a while actually diverge. Besides this, Lagrange pointed out two other defects: the uncertainty as to the exactness of each fresh correction, and the failure to give a commensurable root in finite terms. Lagrange announced (1767) a new method which was free from these defects.[7] It consisted of three parts: (1) a scientific method of finding the integral part of the root instead of the empirical methods hitherto employed; (2) a rule for separating the roots; and (3) a technique of approximation by the use of continued fractions. This last may be explained as follows:

Suppose that in the equation $f(x) = 0$ we have found that there is only one root between the integers p and $p + 1$; for convenience let it be real and positive. Then $x = p + \frac{1}{y}$; and y is greater than unity. Substituting the value of x in $f(x)$ gives an equation $F(y) = 0$.

[7] Lagrange, *Traité*, Chap. I, III, IV and Notes IV, V.

Since $y > 1$, find its integral value by the prescribed method. Suppose it lies between the integers q and $q + 1$. Then $y = q + \frac{1}{z}$; and so on. Then $x = p + \cfrac{1}{q + \cfrac{1}{r + \dots}}$

Lagrange illustrates his method by numerical examples. One of these is Newton's equation $x^3 - 2x - 5 = 0$.[8] By his methods of limits and separation he finds one root between 2 and 3. Let $x = 2 + \frac{1}{y}$; this gives the equation $y^3 - 10y^2 - 6y - 1 = 0$. Again applying the method of limits and separation he finds that y lies between 10 and 11. Similarly for $y = 10 + \frac{1}{z}$ he derives the equation $61z^3 - 94z^2 - 20z - 1 = 0$; here $1 < z < 2$.

Letting $z = 1 + \frac{1}{u}$ gives $54u^3 + 25u^2 - 89u - 61 = 0$: here $1 < u < 2$.

A continuation of this process gives the series 2, 10, 1, 1, 2, 1, 3, 1, 1, 12.

Hence $x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots\dots}}}}$

Evaluating this expression by known methods gives the approximating series: $\frac{2}{1}, \frac{21}{10}, \frac{23}{11}, \frac{44}{21}, \frac{111}{53}, \frac{115}{75}, \frac{576}{275}, \frac{731}{349}, \frac{1307}{624}, \frac{16415}{7837}$, etc.

[8] *Traité* (1798), Chap. IV, p. 29.

That is, $x = \dfrac{16415}{7837} = 2.09455149$; and the error is less than $\dfrac{1}{(7837)^2} = 0.0000000163$.

Though none of these methods gained popular usage, they gave rise to by-products that were incorporated by algebraists into Newton's method. Among these were Rolle's theorem for locating roots and the method of Lagrange for finding the integral parts of the root, as well as his rule for separating roots. Waring had already (1762) formulated such a rule but it was very little known until Lagrange published his method of squared differences.

5. SUMMARY

1. Between 1685 and 1819, three potential rivals of Newton's method sprang into being: Rolle's Method of Cascades (1690), Bernoulli's (1728) and Euler's (1748, 1770) method of recurring series, and Lagrange's method of continued fractions (1767).
2. None of these methods, though scientific and effective, became generally adopted, and Newton's method was supreme from about 1694 to 1819.

IX

HORNER'S METHOD; SIMILAR METHODS BY RUFFINI AND BY THE CHINESE OF THE THIRTEENTH CENTURY

I. A GENERAL VIEW

No method, not even that of Vieta, had been so universally adopted as that of Newton. Of its two latest rivals the method of recurring series, though as presented in Euler's Algebra not particularly laborious, never had its possibilities exhausted; and Lagrange's method of continued fractions, though effective and perfect in theory, was too laborious for popular adoption. At the beginning of the 19th century, therefore, Newton's method held undisputed sway. Its one drawback had been its insecurity. But the success of Waring and Lagrange in effecting methods for the separation of the roots had to a large degree eliminated this defect. The discoveries of Fourier and Budan, early in the nineteenth century, whereby the number and nature of the roots could be detected for any interval, gave Newton's method a much more scientific basis.[1] The mechanism of its operation had been so greatly improved by Halley, Taylor, and Simpson, that in this respect it has to this day remained without a peer.

It has had only one serious rival. It is that form of approximation by the evolution of digits made known to the Western world by William Horner in 1819. Like the methods of Vieta and Newton, it is also a process of exhaustion. Horner's process resembles that of ordinary division or of current methods of finding square and cube roots. That element was not new, having been the chief principle in Vieta's method. But Horner's way of doing this by the gradual diminution of the roots in a succession of transformed equations, was new to Europe. To the world it was old, however; it had been known and used in at least three mathematical centers before 1819.

[1] As early as 1768 J. Raym. Mouraille had successfully formulated a plan for insuring the security of Newton's method; but his work was unknown at the time, and has only recently been noticed. See article by F. Cajori in *Bibl. Math.* (3) XI (1911), pp. 132–37.

—China, Japan, and Italy. In this chapter we shall give a short survey of the different forms this method has assumed.

2. THE METHOD USED IN CHINA AND JAPAN

As early as the first half of the 7th century Wang Hs'iao worked with numerical cubic equations in his *Arithmetic in Nine Sections.*[2] He found the roots by using a method similar to that of finding the cube root of numbers. He attempted no equations of degree higher than the third.

Not till six centuries later do we find any work with equations higher than cubics.[3] Then we find three men, living almost at the same time, systematizing a method for the orderly evolving of the digits of the root. The first, Ch'in Chiu-shao, in his *Nine Sections of Mathematics* (1247), solved equations of the 6th, 7th, 8th, and even higher degrees. He worked on tables, with wooden rods to represent his coefficients, and arranged his work in columns as did the Hindus for roots of numbers. In writing his equations with the right hand member equal to zero he anticipated Harriot by over three hundred years. Li Yeh, the second pioneer in this work, in *The Mirror of the Mensuration of Circles* (1259), explained the process more fully; Ch'in had chiefly given rules. The third member of the trio was Chu-chieh. His form of the process, as presented in his *Introduction to Mathematical Studies* (1299), found its way into Japan in the seventeenth century. The names given to this method are quite suggestive. Ch'in Chiu-shao (1247) called it "Harmoniously alternating evolution"; Yang Hwuy, another writer of the same period, called it "Accumulating involution."

To show the steps in this process, invented in China in the thirteenth century and introduced into Japan in the seventeenth century, we shall give the solution of a problem in which the root is 12, taken from the eighth book of Sātō Moshun's *Tengen Shinan* (1698). The arrangement [4] is Occidental. The solution is as follows:

[2] Yoshio Mikami, *The Development of Mathematics in China and Japan*, New York 1914; pp. 53–56.

[3] D. E. Smith and Yoshio Mikami, *A History of Japanese Mathematics*, Chicago, 1914; pp. 49–56; Yoshio Mikami, pp. 63–89.

[4] Given by D. E. Smith and Y. Mikami. The Chinese arrangement is given in full in Smith and Mikami's *Hist. of Jap. Math.*, pp. 53–55.

Given $x^4 + 4x^3 - 236x^2 - 432x + 11520 = 0$

$$
\begin{array}{rrrrr}
1 & +\ 4 & -\ 236 & -\ 432 & +\ 11520 \\
 & 10 & 140 & -\ 960 & -\ 13920 \\
\hline
1 & 14 & -\ 96 & -\ 1392 & -\ 2400 \\
 & 10 & 240 & 1440 & \\
\hline
1 & 24 & 144 & 48 & -\ 2400 \\
 & 10 & 340 & & 2400 \\
\hline
1 & 34 & 484 & 48 & 0 \\
 & 10 & & 1152 & \\
\hline
1 & 44 & 484 & 1200 & \\
 & 2 & 92 & & \\
\hline
1 & 46 & 576 & &
\end{array}
$$

Among Japanese scholars to write on the Chinese method [5] of approximation were Sātō Seikō (1666), Seki Kōwa (1642–1708), Sātō Moshun (1698), Sakabe (1803), and Kawai (1803).[6]

3. THE METHOD INVENTED BY RUFFINI

In 1804 a solution very similar to the approximation process of the Chinese was effected by Paolo Ruffini.[7] By means of the differential calculus he worked out a theory for transforming one equation into another with roots diminished by a certain constant. In his revisions of 1807 and 1813 he used algebraic processes for the same purpose. His arrangement differs from that of Horner in that the coefficients of the transformed equation appear in the extreme right hand column, whereas in Horner's arrangement they appear in a diagonal line. A single solution will tend to show the similarity of Ruffini's work to Horner's process and to that of the Chinese:

[5] Whether the process used by the Chinese was actually the same as that used by Horner is questioned by Professor Gino Loria, of the University of Genoa, Italy, in an article published in the *Bolletino della Mathesis* for April, 1920, and abridged by Professor R. B. McClenon, of Grinnell College, Grinnell, Iowa, in *The Scientific Monthly* for June, 1921. In discussing Ch'in Chin-Shao's solution of the equation $-x^4 + 763,200x^2 - 40,642,560,000 = 0$ he says: "We still have too meager information as to the details of the work for us to be able to affirm confidently that Horner's method was known to the Chinese in the 13th century; we can only say that this method, or one practically identical with it, was known at the time, and we must await further evidence before affirming or denying the priority of the Chinese in its discovery."

[6] Smith and Mikami, pp. 53, 86, 116, 212–15.

[7] F. Cajori, "Horner's Method of Approximation Anticipated by Ruffini," in *Bull. American Math. Soc.*, XVII (1911), pp. 409–414.

"Given the equation $4x^5 - 6x^4 + 3x^3 - 5x^2 - 4x + 8 = 0$; required, to transform it into another in which $y = x - 6$. Operate in the manner here set forth:

$$\begin{array}{rrrrrr}
4, & -6, & 3, & -5, & -4, & 8, \\
 & 4, & 18, & 111, & 661, 3962, & 23780 \\
 & & 4. & 42, & 363, 2839, & 20996 \\
 & & & 4, & 66, \ 759, & 7393 \\
 & & & & 4, \ \ 90, & 1299 \\
 & & & & 4, & 114 \\
 & & & & & 4
\end{array}$$

and we obtain the transformation

$$4y^5 + 114y^4 + 1299y^3 + 7393y^2 + 20996y + 23780 = 0."$$

For Newton's classic example $x^3 - 2x - 5 = 0$ he finds the root to 8 decimal places, by this method.

Ruffini's method remained unnoticed by his own people. It was the story of Vieta repeated. But there was no Harriot or Oughtred to bring the message to England this time. For Britain had her own discoverer in this field. It is a coincidence that every successful approximation method for numerical equations has been either invented or sponsored in Great Britain.

4. HORNER'S METHOD

The theory and mechanism of Horner's method are so well known that we need not discuss them here.[8] We shall review only the outstanding points in its history.

It is safe to say that neither Horner nor Ruffini knew of the other's work, nor was either acquainted with the methods of the Orient. Horner's paper was read before the Royal Society of London on July 1, 1819, and was published in the *Philosophical Transactions* for that year.[9]

In developing his theory, he used Taylor's Theorem and Arbogast's derivatives (Ruffini had used ordinary derivatives), and

[8] A thorough discussion of its analytic and operational phases is to be found in J. R. Young's *Theory and Solution of Algebraical Equations of Higher Order*, London, 1843, Chapter XII. A vigorous challenge to rival English claimants for priority over Horner is to be found in De Morgan's article already referred to.

[9] W. H. Horner, "A new method of solving numerical equations of all orders by continuous approximation," in *Phil. Trans.*, Vol. 109 (1819), pp. 308–335.

arranged his results in a "General Synopsis" in much the same way as Halley built up his General Analytical Speculum. He obtained his transformations by using the calculus, but says that algebraic processes can be used for this purpose. Like Ruffini he used ordinary algebra in his revisions of 1830 and 1845.

His first illustration is the classic example of Newton. The work is arranged differently from that which we see in text-books on algebra. His *schema* resembles Wallis's arrangement of Vieta's process and our present arrangement in finding the cube root; he even divides the resolvend into periods.

Horner's method met with quick response in England, due in part to its enthusiastic advocates, J. R. Young and A. De Morgan. It is today a strong rival to Newton's method in both England and the United States. It is widely used, though less than Newton's method, in Germany, Austria, and Italy. In France, Newton's method has held undisputed sway. Horner's method, when combined with the theorems of Budan and Fourier and of Sturm, is effective, secure, and expeditious.

5. SUMMARY

1. In the 13th century Ch'in Chiu-shao (1247), Li Yeh (1259), and Chu-chieh (1299) employed a method of approximation virtually the same as Horner's method. In the 17th century it was introduced into Japan.
2. Paolo Ruffini (1804) invented a similar method in Italy, which was soon forgotten.
3. Horner published his method in 1819, and it soon became widely used in England and later in the United States, and to a less degree in Germany, Austria, and Italy. It was never adopted in France.

X

GENERAL SUMMARY

1. The earliest known method of finding the roots of numbers and of solving formal equations is that of Single False Position. This method was used by the Egyptians, the Greeks, and the Hindus.

2. The method of averages was probably used by Archimedes; a definite exposition of it is given by Heron of Alexandria. Its applicability was restricted; but it was often incorporated into other methods, as in Chuquet's Rule of Mean Numbers.

3. Methods of Double False Position for finding the roots of numbers seem to have been used by Archimides; a specific process was described by Heron of Alexandria. In algorithmic work double false position was used by the Arabs and Leonardo of Pisa. By the Renaissance writers these processes were extended to the solutions of numerical higher equations. Though eclipsed by the methods of exhaustion invented by Vieta, Newton, and Horner, methods of double false position have never become obsolete. Two main plans for finding mean values have been followed: (1) the averaging or analogous devices, as illustrated by Chuquet and DeLagny; (2) interpolation by proportional parts as used by Leonardo of Pisa, al-Karchî, Bürgi, and Pitiscus.

4. Methods of exhaustion have been used the most extensively of all approximation methods. The earliest method of exhaustion recorded is that used by Theon of Alexandria for irrational square roots by reversing the binomial formula. By a special adaptation of this method Āryabhaṭa, Brahmagupta, Śrīdhara, and Bhāskara worked out a process for evolving the roots of numbers digit by digit (Evolution). This was adopted by the Arabs and by them transmitted to Christian Europe through the writings of Leonardo of Pisa and of Sacrobosco. Our present standardized process of finding roots of numbers is largely the work of Peurbach and Cardan.

Vieta extended this method to numerical higher equations. His method was popularized and made more practicable by Harriot, Oughtred, and Wallis, and remained the prevailing method until superseded by Newton's method.

5. Newton's method of exhaustion, based on equations whose

terms constitute an infinite series, was conceived as an aid to his work in the calculus. The place of this method as a constituent part of algebra is largely due to its modification and systematic development by Raphson. Its technique was improved by DeLagny, Halley, Taylor, and Simpson and its scientific basis was clarified and strengthened by Lagrange, Fourier, Budan, and Sturm. No other method of approximation has come up to it in general popularity.

6. A third method of exhaustion, employing continued fractions, was invented by Lagrange. It is a theoretically perfect method, but too laborious for practical use. It has retained chiefly an academic interest.

7. Rolle's Method of Cascades and the method of recurring series developed by Bernoulli and Euler have been used only slightly. The methods invented by Collins, Fontaine, and some of those used by DeLagny were never taken up into the mathematical activities of the world.

8. A fourth method of exhaustion in solving numerical higher equations is both the oldest and the youngest,—oldest, because it was used by the Chinese in the thirteenth century, youngest, because it was re-invented and given to the Western world by Ruffini and Horner in the twentieth century. It is a process of evolution of digits resembling that of Vieta; however, the digits are evolved, not by the inversion of a binomial formula, but by the transformation of equations. Horner's method has made progress in Italy, Germany, and Austria, but is hardly ever used in France; in England and the United States it is a close rival of Newton's method.

APPENDIX

Note 1

The solution of numerical higher equations by means of graphs will often give as near an approximation as is desired.[1] Where greater accuracy is desired a combination of the graphic method with one of the standard methods of exhaustion or of double false position is generally advantageous. The application of the calculus, such as is ordinarily used in curve tracing, facilitates the work of approximation; in isolating the roots, for example, one can often use Rolle's theorem instead of the more elaborate theorems of Fourier, Budan, and Sturm.

Whether tables will ever be used in the solution of numerical equations is for the future to tell. DeLagny advocated their use and worked out a comprehensive set of tables similar to our modern integration tables in his *Analyse Générale;* but, as his voluminous work has been very little known, his idea has never been tried out.

Note 2

Teachers of algebra often fail to recognize how extensively the methods of double false position are used in mathematical thinking. Notice how similar to Bürgi's method of solving equations is the following common method of determining the autumnal equinox: As given in the Almanac of 1899, the sun's declination at approximate noon (Greenwich), Sept. 22, was $+ 0° 18' 8''.7$ and at noon, Sept. 23, it was $- 0° 5' 13''.9$. The difference in declination being $23' 22''.6$, the equinox is found, by interpolation, to be $18' 18''.7 \div 23' 22''.6$ or 0.7762 of a day after noon, Sept. 22. Take also the *Law of Charles* in physics, by which we know that if the temperature of a gas be lowered 1° C. its volume will be decreased by $\frac{1}{273}$ of its volume at 0° C., and if its temperature is lowered 10° C. its volume will decrease by $\frac{10}{273}$ of its volume at 0° C. From these two positions the volume at any temperature, say − 100° C., for which the gas does not become a liquid, is calculated. The Fahrenheit-Centigrade thermometer problems, the interpolations for trigonometric functions, and many problems in proportion are more familiar illustrations.

[1] Consult Arthur Schultze, *Graphic Algebra*, New York, 1908.

The formal and informal interpolation processes used in astronomy, physics, trigonometry, and other computational fields outside and inside the classroom constitute a big share of the actual calculation. There seems to be no valid reason, then, why the student of algebra should not have his attention called to the use of such methods in the solution of numerical higher equations. While less refined than methods of exhaustion, they are very usable for securing results correct to one or two decimal places. An interesting textbook exposition is that given by Professor Isaac Dalby in the second volume of his *A Course of Mathematics, designed for the use of the Officers and Cadets of the Royal Military College* (London, 1806), pp. 175–185. For good illustrations of the use of these methods in recent texts, see Wells's *University Algebra* (Boston, 1884), pp. 417–19, and F. L. Griffin's *Introduction to Mathematical Analysis* (New York, 1921), pp. 37–38.

To find the mean between the two false positions we generally use one of the following three ways: (1) finding the arithmetic average, or using some such special formula as Chuquet's Rule of Mean Numbers; (2) interpolating directly by proportional parts; (3) using the interpolation formula

$x = \dfrac{a_2e_1 - a_1e_2}{e_1 - e_2}$, where a_1 and a_2 are the approximate values and e_1 and e_2 are the corresponding errors.

Note 3

The solutions of numerical higher equations first became a mathematical need in the work of geometry, trigonometry, and astronomy. However, they enter into expert practical calculation more frequently than the average student realizes, and the author feels convinced that the insertion of such problems would vitalize the methods of approximation given in our college algebras. The following representative problems from the fields of physics, mechanics, economics, and engineering, selected from the more progressive text-books, will serve to illustrate:[2]

(1) The real root of the equation $x^5 - 38x - 101 = 0$ gives the diameter of a water pipe 200 feet long, discharging 100 cubic feet per second under a head of 10 feet.

Result: diameter is 2.92 ft.

(2) The equation $v^3 - 19200v + 211200 = 0$ gives the speed in feet per second on a 1-inch manila rope transmitting 4 horse-power, under a tension of 300 pounds on the tight side.

[2] Merriman and Woodward, *Higher Mathematics*, p. 13; Rietz and Crathorne, *College Algebra*, p. 140; Karpinski, Benedict, and Calhoun, *Unified Mathematics*, p. 404; F. L. Griffin, *Introduction to Mathematical Analysis*, pp. 37–42, 337–342. More problems of a similar nature may be found in these same references.

Result: speed is 11.07 feet.

(3) From the American Report on Wholesale Prices, Wages, and Transportation, for 1891, the median wage in dollars is given by $\frac{1}{4}$ of a value of x in the equation $2561\frac{1}{2} = 6972\frac{19}{128} + 657\frac{5}{18}x - 33\frac{43}{64}x^2 + \frac{197}{48}x^3 - \frac{5}{128}x^4$.

Result: wages were \$1.536.

(4) In Archimedes's problem of the section of a sphere, if k represents the ratio of the larger to the smaller segment, the distance of the plane section from the center of a sphere 10 inches in diameter is given by the equation

$$x^3 - 300x + 2000\left(\frac{k-1}{k+1}\right) = 0.$$

(5) A sphere of yellow pine 1 foot in diameter floating in water sinks to a depth x given by the equation $2x^3 - 3x^2 = 0.657$.

Result: depth is 0.606 foot.

(6) The equation $d^4 + 320d^2 - 340d - 4290 = 0$ gives the smallest safe diameter (in inches) for a certain steel shaft.

Result: the diameter is 4.11 inches.

(7) The equation $2x^3 - 9x^2 + 24 = 0$ gives the depth of immersion in water of a sphere 3 feet in diameter and with a specific gravity $\frac{8}{9}$.

Result: depth is 2.38 feet.

(8) The equation $x^4 - 118x^2 + 160x - 115 = 0$ gives the length of the longest rectangular panel 1 foot wide which can be fitted diagonally across a door 4 feet wide and 10 feet long.

Result: length is 10.17 feet.

(9) The "index of correlation" between the eye-colors of a certain group of people and of their great-grandparents is approximately the root of the equation $0.024x^4 + 0.137x^3 + 0.035x^2 + x - 0.225 = 0$ which lies between 0 and 1. [C. B. Davenport]

Result: index is 0.22.

(10) Using Jupiter's radius, 45090 miles, as the unit of distance, the equation $x^3 - 5x^2 + 6.27396x - 0.060385 = 0$ gives the greatest and least distances of Jupiter's Fifth Satellite from the center of the planet.

Results: distances are 2.48 and 2.51 units.

(11) A magnet placed with its ends in a "magnetic meridian" will neutralize the earth's magnetism at certain points. To calculate the position of these points in a certain case, it was necessary to solve the equation $\frac{20000x}{(x^2 - 100)^2} = 0.2$.

Result: $x = 47.82$ and 0.09.

Note 4

Bearing upon source material relating to the solution of numerical equations, the reader may be interested to know that in New York libraries are found certain works very rare in this country. Among these may be mentioned the manuscript copies of the *Lilawati*, the *Kholasat-al-Hisâb*, and of Chuquet's *Triparty* in the private collection of Professor David Eugene Smith. In the Columbia University library there is a manuscript on algebra by Scheubel, in his own handwriting, prepared for the press but never published; also a copy of Vieta's collected works published by van Schooten in Leyden in 1646, and a copy of J. Hume's exposition of Vieta's algebra, published in Paris in 1636. The latter work seems to have been unknown to the mathematical world, as we find no references to it in any published history of mathematics.[3] Hume anticipated the "Newtonian divisor" by an analogous device, but his work was evidently unknown to his contemporaries. In the New York Public Library are found three comparatively rare books: Hérigones' *Cursus Mathematicus* (Paris, 1642), which popularized Vieta's method in France; Michel Rolle's *Démonstratio d'une Méthode;* and Raphson's *Analysis Aequationum Universalis*, edition of 1697, bound with his almost wholly unknown metaphysical treatise, *De Spatio Reali*.

[3] It is referred to in Professor D. S. Smith's forthcoming history of mathematics.

VITA

Martin Andrew Nordgaard was born January 29, 1882, near Northwood, Iowa. He graduated from Northwood High School in 1898, and in 1903 he received his Bachelor of Arts degree from St. Olaf College, Northfield, Minnesota. From 1903 to 1909 he served as teacher and administrator in the secondary schools of Wisconsin and Minnesota, with the exception of one year spent in the study of education at the University of Minnesota. He was the head of the department of mathematics at Columbia Lutheran College in Everett, Washington, from 1909 to 1912. The following year he was a graduate student of mathematics at the University of Chicago. The three years from 1913 to 1916 he was instructor of mathematics at the University of Maine. From this institution he received the degree of Master of Arts in 1914. During the year 1916–17 he served as interim head of the department of education at St. Olaf College. In 1917–18 he held the Harrison scholarship in mathematics in the graduate school of the University of Pennsylvania. In the fall of 1918 he became a member of the mathematics staff of Grinnell College, Grinnell, Iowa. In February, 1921, he was granted leave of absence for the purpose of further study. Since then he has been a graduate student in Teachers College, Columbia University, majoring in the department of mathematics. During the current year he has held the position of lecturer in mathematics at Columbia University.